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Short communication: Forward and inverse analytic models relating river long profile to tectonic uplift history, assuming a nonlinear slope–erosion dependency

Yizhou Wang et al.

Correspondence to: Yizhou Wang (wangyizhou2016@outlook.com)

Text S1: Concave-up knickpoint migration (decreasing *U* and *n* < 1)

Figure S1a shows the evolution of a channel profile under $n < 1$ and a step-decrease in the tectonic uplift rate from U_0 to U_1, leading to the migration of a concave-up knickpoint (Figure S1b). Similar to the model shown in Figure 1, channel segments below and above the knickpoint are in equilibrium with respect to U_1 and U_0, respectively. Figure S1b shows the change in knickpoint elevation when it migrates from point A to D, during time interval *dt*. The elevation difference, segment DG, can be expressed as:

$$\mathrm{DG} = z_{t+dt}(x + dx) - z_{t+dt}(x) = (\frac{\partial z}{\partial x})_1 \cdot v_\mathrm{H} \cdot d, \quad \text{(S1)}$$

where v_H is the knickpoint celerity. Using the geometric relationship:

$$\mathrm{DG} = \mathrm{BG} - \mathrm{BD}, \quad \text{(S2)}$$

$$\mathrm{BD} = -(U_1 - U_0) \cdot d, \quad \text{(S3)}$$

$$\mathrm{BG} = (\frac{\partial z}{\partial x})_0 \cdot v_\mathrm{H} \cdot d, \quad \text{(S4)}$$

Equations (S1-S4) can be combined to derive v_H:

$$v_\mathrm{H} = \frac{(U_1 - U_0)}{\left(\frac{\partial z}{\partial x}\right)_1 - \left(\frac{\partial z}{\partial x}\right)_0}, \quad \text{(S5)}$$

Equations (13) and (S5) are identical, indicating that the celerity of concave-up knickpoints formed in response to decreasing *U* and $n < 1$ and convex-up knickpoints formed in response to increasing *U* and $n > 1$ is the same.

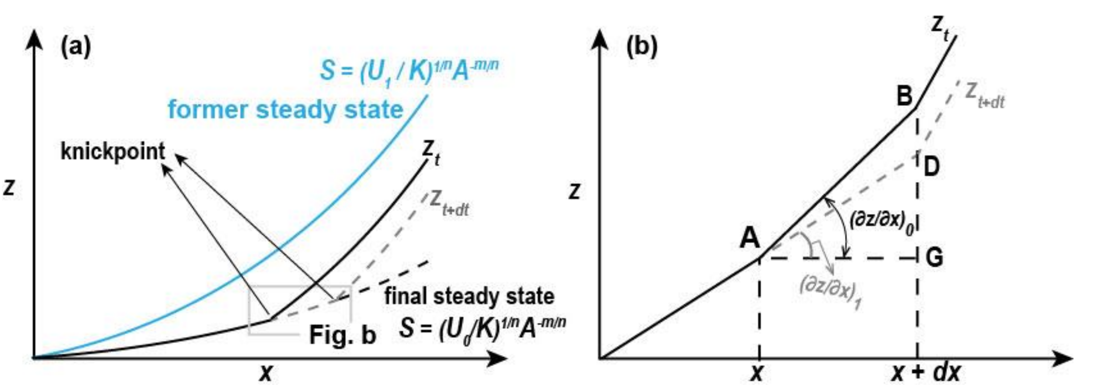

Figure S1: (a) The adjustment of channel profile to a decrease in the tectonic uplift rate from U_0 to U_1. (b) Knickpoint (concave-up) migration under deceasing *U* and $n < 1$. Knickpoint retreats from point A to D during time interval *dt*, with a transition of channel profile from z_t to z_{t+dt}.

Text S2: Theoretical comparison between the model of kinematic wave speed and slope-break knickpoint retreat

S2.1 Difference in estimating knickpoint age

We compare and contrast the slope break migration model derived here to the kinematic wave speed model (e.g. Rosenbloom and Anderson, 1994; Weissel and Seidl, 1998; Oskin and Burbank, 2007; Royden and Perron, 2013). In the wave speed model, the horizontal speed for knickpoint migration ($v_{\mathrm{H\_Ro}}$) is:

$$v_{\mathrm{H\_Ro}} = K \cdot A^{m} \cdot S^{n-1}, \tag{S6}$$

The linear form (n = 1) has been widely used to calculate knickpoint age and infer regional incision/uplift history (Goren et al., 2022).

For the non-linear case, when the slope exponent, $n \neq 1$, the kinematic wave speed depends on the channel gradients, with either positive (n > 1) or negative (n < 1) correlation. Local variations in channel gradients may cause changes to the shape of the kinematic wave and potentially alter knickpoint recession rate (Whitham, 1974; Berlin and Anderson, 2007). Numerical implementation of equation (S6) is not trivial, because the slope, S, at the slope-break is ill defined. Indeed, Whipple and Tucker (1999) pointed out that the kinematic wave speed, equation (S6), and its derivations are valid for infinitesimal, step perturbations (e.g., coseismic, eustatic, or stream capture), rather than for sustained changes in uplift rates.

Assuming a small perturbation in the uplift rate, which causes no change in the channel steepness index, the channel profile below the knickpoint is in equilibrium with the tectonic forcing (Whipple and Tucker, 1999), and equation (3) could be assigned in equation (S6):

$$v_{\mathrm{H\_Ro}} = KA^{m}\left(k_{\mathrm{s\_1}} A^{-m/n}\right)^{n-1} = K k_{\mathrm{s\_1}}^{n-1} A^{m/n}, \tag{S7}$$

The fluvial response time of the kinematic wave knickpoint could be expressed as (Whipple and Tucker, 1999):

$$\tau_{\mathrm{Ro}} = \int_0^{x_p} \frac{1}{k_{\mathrm{s\_1}}^{n-1}} \cdot \frac{1}{KA(x)^{m/n}}\, dx = \frac{1}{KA_0^{m/n}} \frac{1}{k_{\mathrm{s\_1}}^{n-1}} \chi(x_p), \tag{S8}$$

where x_{p} defines the present location of the knickpoint along the river long profile. Comparing equations (14) and (S8), the ratio in the response time of the two models, the slope-break model developed here and kinematic wave recession model, can be expressed as:

$$\frac{\tau}{\tau_{\mathrm{Ro}}} = \frac{\frac{1}{KA_0^{m/n}} \cdot \frac{k_{\mathrm{s\_1}}(1-\gamma_{0\_1})}{k_{\mathrm{s\_1}}^{n}(1-\gamma_{0\_1}^{n})} \chi(x_p)}{\frac{1}{KA_0^{m/n}} \cdot \frac{1}{k_{\mathrm{s\_1}}^{n-1}} \chi(x_p)} = \frac{1-\gamma_{0\_1}}{1-\gamma_{0\_1}^{n}}, \tag{S9}$$

Hence, we agree with Whipple and Tucker (1999) that, in the general case, the kinematic wave recession model is an inaccurate description of knickpoint formed following a sustained change in the rate of tectonic rock uplift. Interestingly, the two models become identical for $n = 1$.

S2.2 The dependence of knickpoint retreat distance on catchment drainage area for both knickpoint migration models

The integral in equation (S8) defines an enclosed area below the function $A(x)^{-m/n}$ and within the interval of (0, x_p). By the mean value theorem of integrals, a rectangle with an equal area can be defined, that has a length of x_p and a width equal to $A(x_1)^{-m/n}$ for $0<x_1\leq x_p$. This allows the following transformation of equation (S8):

$$x_p = K\tau_{\mathrm{Ro}} k_{\mathrm{s\_1}}^{n-1} A(x_1)^{m/n}, \tag{S10}$$

Equation (S10) predicts a power-law scaling between knickpoint retreat distance, x_p, and the drainage area at x_1. Estimate x_1 can be done by assuming Hack's law (Hack, 1957):

$$A(x) = k_a(L-x)^h, \tag{S11}$$

where k_a and h are positive constants, L is total length of the river, and x measures the distance from the outlet. Assigning equation (S11) into the term, $\chi_p/A_0^{m/n}$, we can derive:

$$\frac{1}{A_0^{m/n}}\chi_p = \int_0^{x_p} \frac{1}{k_a^{m/n}(L-x)^{hm/n}}\,dx = \frac{1}{k_a^{m/n}(L-x_1)^{hm/n}}\,x_p, \tag{S12}$$

Thus, the position of x_1 can be derived from:

$$\frac{A_0^{m/n}}{k_a^{m/n}} \cdot \frac{x_p}{\chi_p} = (L-x_1)^{hm/n}, \tag{S13}$$

Since Hack's law is an approximation to the drainage area distribution, we do not aim to calculate the accurate position of x_1. More significantly, equation (S13) indicates a positive control of river total length, L, over knickpoint retreat distance, x_p. Thus, we show that (1) the models of kinematic wave and slope-break knickpoint retreat share the same mathematical form only when $n = 1$ and (2) catchment drainage area, determined by channel total length (equation S11), is key in controlling how far the knickpoint (both a sudden base level fall and a sustained increase in tectonic uplift rate) has moved upstream. Similar results are attained in statistic models such as in Castillo et al. (2017).

Text S3: Landscape evolution for the simple case of one single knickpoint

Here, we present the resemblance between the current derivation (equations 14 and 21) and equations 5c and 6c in Mitchell and Yanites (2019), respectively. Our equation (14) expresses the response time (age) of a knickpoint:

$$\tau(x_p) = \frac{k_{\mathrm{s\_1}}(1-\gamma_{0\_1})}{k_{\mathrm{s\_1}}^n(1-\gamma_{0\_1}^n)} \cdot \frac{1}{KA_0^{m/n}} \cdot \chi(x_p), \text{ with } \gamma_{0\_1} = k_{\mathrm{s\_0}}/k_{\mathrm{s\_1}} \tag{S14}$$

Using the relation, $U_0 = K \cdot k_{s\_0}^n$ and $U_1 = K \cdot k_{s\_1}^n$, we can write:

$$\tau(x_p) = \frac{k_{s\_1} - k_{s\_1}}{k_{s\_1}^n - k_{s\_0}^n} \cdot \frac{1}{K A_0^{m/n}} \cdot \chi(x_p) = \frac{(U_1/K)^{1/n} - (U_0/K)^{1/n}}{U_1 - U_0} \cdot \frac{1}{A_0^{m/n}} \cdot \chi(x_p) \quad \text{(S15)}$$

Upon rearrangement:

$$\chi(x_p) = \frac{\tau(x_p) \cdot (U_1 - U_0)}{(U_1/K)^{1/n} - (U_0/K)^{1/n}} A_0^{m/n} \quad \text{(S16)}$$

Which is identical to equation 5c of Mitchell and Yanites (2019).

Adopting the assumption of Mitchell and Yanites (2019) of a single step increase in the uplift rate, from U_0 to U_1 ($U_1 > U_0$), equation (21) in our study can be simplified to be:

$$z\left(t, x_p(t)\right) = U_1 t + \left[\frac{(1-\gamma_{0\_1}^n)}{(1-\gamma_{0\_1})} - 1\right] \cdot U_1 \cdot t = \frac{1-\gamma_{0\_1}^n}{1-\gamma_{0\_1}} \cdot U_1 \cdot t$$

where $\gamma_{0\_1} = k_{s\_0}/k_{s\_1} = (U_0/K)^{1/n}/(U_1/K)^{1/n} = (U_0/U_1)^{1/n}$. Thus, we can write:

$$z\left(t, x_p(t)\right) = \frac{1 - U_0/U_1}{1 - (U_0/U_1)^{1/n}} \cdot U_1 \cdot t = t \cdot (U_1 - U_0) \cdot \frac{{U_1}^{1/n}}{{U_1}^{1/n} - {U_0}^{1/n}}$$

This equation is similar to equation 6c of Mitchell and Yanites (2019). Importantly, our equation (21) is more general, as it does not assume only a single step change in *U*.