Advection and dispersion of bedload tracers : response to referees

E. Lajeunesse, O. Devauchelle, and F. James

We thank the referees for their careful reading of our paper, and for the issues they raised. They will find below our answers, and the changes made to the text. For the sake of clarify, we reproduce the referees's comments in italics, followed by our answers. Similarly, the changes applied to the article appear in blue on the new manuscript, uploaded as supplementary material.

I. RESPONSE TO REFEREE 1

A. General comments

This paper expands on a previously published model for describing fluvial bedload tracer transport. The model is derived from a mass balance based primarily on the erosion and deposition rates and the fractions of tracer in the mobile and immobile parts of the bed. The model assumes a homogenous bed grain size and steady and uniform transport. From the model, the authors derive analytic expression for the velocity and dispersion of a tracer plume. The paper is well organized and the derivations are mostly clear, with the exception of the derivation of the scaling of the mean and variance of the tracer plume with time. My difficulty following this derivation is perhaps the origin of my most serious concern, discussed below.

a *My most serious concern is the prediction of how the mean and variance scale with time t during the entrainment regime, specifically that the mean increases as* t^2 *and the variance as* t^3 *(page 6, figure 3, and equations 37-39). I am not familiar with any literature that describes variance scaling greater than the ballistic regime, variance* $\sim t^2$*. The Zhang et al. [2012] and Nikora et al. [2002] references cited on page 6, line 15 do not support this variance scaling. Weeks and Swinney [1998] Figure 1 shows an upper limit to the variance scaling of* t^2 *and Table 1 limits the scaling of the mean to* $t^x, x < 2$*. All the observations of anomalous super-diffusion with which I am familiar [e.g. Bradley, 2017; Phillips et al., 2013] show a variance scaling of* t^x *with* $1 < x < 2$*. The paper needs more discussion of how these predictions are consistent with previous theoretical work and observations of tracer dispersion.*

As the referee points out, super-diffusion cannot lead to variance increasing faster than $\sim t^2$. Our claim that the entrainment regime involves super-diffusion was incorrect. In fact, the scalings derived for the entrainment regime show that it is of a different nature than super-diffusion. We have changed several paragraphs in sections 3, 5, 6, and in the conclusion : page 7- lines 30-33, and page 15 - line 26. The new manuscript also includes the following discussion, stressing out the difference between anomalous diffusion and the entrainment regime (page 12 lines 21 - page 13 line 7) :

> "Anomalous diffusion arises from heavy-tailed distributions, of either the step length or the waiting time (Weeks and Swinney, 1998). The erosion-deposition model contains no such ingredient. Here the fast increase of the variance results from the exchange of particles between the sediment bed and the bedload layer, at the beginning of the experiment. Over a time shorter than the flight duration τ_f, the tracers entrained by the flow do not settle back on the bed. They form a thin tail, which leaves the main body of the plume, and moves downstream at the average particle velocity V (Fig. 2 a). The plume therefore consists of a main body of virtually constant concentration, followed by a thin tail of length $\propto Vt$.

> Accordingly, we can split the integral that defines its mean position, equation (12), into two terms. The first one, obtained by integrating cx over the main body of the plume, yields the initial position of the plume $\langle x \rangle_0$. The second one, obtained by integrating cx over a tail of length Vt, scales as t^2. Summing these contributions yields equation (39). Similar reasonings yield equations (40) and (41) for the variance and the skewness."

b *A secondary concern is the connection of the model to the classical advection diffusion equation.*

First, it is unsurprising that the model is equivalent to the ADE because the assumptions in the model do not allow for anomalous dispersion that arises from heavy-tailed step lengths or resting times. For example, the assumption of steady and uniform transport without storage in the bed or bars pre- supposes this outcome and limits the model's applicability to real rivers.

As discussed above, the erosion-deposition model contains none of the ingredients necessary for super-diffusion. Following the referee's comment, we have included in the new manuscript a discussion about the physical mechanism at work in the advection-diffusion regime (page 10 - lines 9-13) :

> "We interpret this formal derivation as follows. In the reference frame of the plume, a tracer at rest on the bed moves backward, while a tracer entrained in the bedload layer moves forward. At long time, the proportions of tracers in each layer equilibrate. Consequently, the probability that a tracer be entrained and move forward, equals that of deposition. In the reference frame of the plume, the exchange of particles between the bed and the bedload layer is thus a Brownian motion, hence the linear diffusion of the plume."

Second, I don't see how the advection velocity U and the diffusion coefficient D are related to α. α is the ratio of mobile concentration to stationary concentration and it is dimensionless. It's not clear how the authors get from a dimensionless quantity to U with dimensions of L/T and D with L2/T.

As of section 3, all equations are dimensionless. Accordingly, the expression of the advection velocity U and the diffusion coefficient D derived at the end of section 4 are dimensionless. With dimensions, they read :

$$U = \frac{\alpha}{\alpha + 1} \, \frac{\ell_f}{\tau_f} \tag{1}$$

$$D = \frac{\alpha}{(\alpha + 1)^3} \, \frac{\ell_f^2}{\tau_f} \tag{2}$$

We have edited the corresponding equations at the end of section 4 to clarify this point.

My final concern is the use of the mean travel distance as a proxy for transport time as a way to account for the intermittency of transport. While this somehow reduces the variance scaling during the entrainment regime to what Weeks and Swinney [1998] define as the allowable range, equating travel distance with time implies a steady tracer virtual velocity and therefore a linear increase of mean tracer position with time. This appears to be inconsistent with the predicted increase in mean position with time (eq. 37) during the entrainment regime. Further discussion of this apparent inconsistency and justification of travel distance as proxy for time is warranted.

We dot not equate travel distance with time. As the referee points out, this would be inconsistent with equation (37) (and several others). Instead, we note that setting the expressions for the mean, variance, and skewness against each other removes time from the problem. Combining equations (37), (38) and (39) even provides analytical expression of the variance and skewness as a function of traveled distance during the entrainment regime. However, interpreting the resulting scaling as a signature of anomalous diffusion was wrong. As discussed above, the entrainment regime is not a super-diffusive regime. We thank the referee for pointing out this inconsistency. We changed section 6 accordingly.

B. Specific comments

The author states on page 2, line 14 that at long times, tracer dispersion is normal, with linearly increasing variance as if it were settled science. In my opinion, this is not a settled issue. Recently published work [Bradley] presents evidence of anomalous super-diffusion over 9 years of observation.

We changed the corresponding paragraph (page 2, lines 11-25) :

> "The dispersion of the tracers, expressed as the variance of their location, results from the randomness of bedload transport. Nikora *et al.* (2002) identify three regimes with distinct time scales. A particle entrained by the flow repeatedly collides with the bed (Lajeunesse *et al.*, 2017). At short time, between two collisions, particles move with the flow, and the variance increases as the square of time (Martin, Jerolmack, and Schumer, 2012; Fathel, Furbish, and Schmeeckle, 2016). This regime is analogous to the ballistic regime of Brownian motion (Zhang, Meerschaert, and Packman, 2012; Fathel, Furbish, and Schmeeckle, 2016).
>
> As the particle continues its course, collisions deviate its trajectory. In this intermediate regime, the variance increases non-linearly with time (Martin, Jerolmack, and Schumer, 2012). Nikora *et al.* (2002) attribute this behavior to anomalous super-diffusion ; but Fathel, Furbish, and Schmeeckle (2016) contest their interpretation.
>
> With time, tracers settle back on the bed, where they can remain trapped for a long time. How the distribution of resting times influences the long-term dispersion of tracers remains unknown. The data collected by Sayre and Hubbell (1965) are consistent with the existence of a diffusive regime, in which the variance increases linearly (Zhang, Meerschaert, and Packman, 2012). Other investigators, however, report either subdiffusion or super-diffusion (Nikora *et al.*, 2002; Bradley, 2017). These anomalous diffusion regimes are sometimes modeled with fractional advection-dispersion equations (Schumer, Meerschaert, and Baeumer, 2009; Ganti *et al.*, 2010; Bradley, Tucker, and Benson, 2010). "

Page 4, line 11. The connection between surface grain size concentration and grain size needs clarification. This seems to imply exactly 1 tracer per unit area, an unrealistic assumption.

The sediment bed is made of uniform particles of size d_s. Each of them occupies an area which scales like $\sim d_s^2$. The surface concentration of particles at rest on the bed surface is therefore $n_s \sim 1/d_s^2$. The concentration of moving particles is much smaller, $n \sim \alpha/d_s^2$. We have edited the corresponding paragraph to clarify this point (page 4, lines 9-10).

Page 5, line 15. Phi = 0 is not the same as phi is null. I assume that the authors meant "nil." Null means undefined and not equal to anything. You can never state x = null.

This sentence is now : "As a result, the proportion of mobile tracers vanishes ($\phi = 0$), and the total concentration of tracers reads $c = \psi/(\alpha + 1)$."

Page 5, line 28-29. The statement that tracers rarely move during the first flood is incorrect and is inconsistent with the statement about tracer installation on the bed surface at the beginning of this paragraph. Nearly all tracer studies *neglect the first episode of transport precisely because tracers placed on the bed surface are unnaturally mobile until they are thoroughly mixed into the bed.*

We have edited the corresponding paragraph to clarify this point (page 6 - lines 7-15) :

> "The early evolution of the plume depends on initial conditions. In most field experiments, tracers are deposited at the surface of the river bed when the flow stage is low and sediment are motionless (Phillips, Martin, and Jerolmack, 2013). During floods, the river discharge increases and the shear stress eventually exceeds the entrainment threshold, setting in motion some of the grains. The entrainment of particles strongly depends on the arrangement of the bed : grains highly exposed to the flow move first (Charru, Mouilleron, and Eiff, 2004; Turowski, Badoux, and Rickenmann, 2011; Agudo and Wierschem, 2012). Several authors find that the tracers they disposed on the bed are more mobile during the first flood than during later ones (Bradley and Tucker, 2012). During the later floods, tracers gradually get trapped in the bed, and their average mobility decreases. On the other hand, Phillips and Jerolmack (2014) find no special mobility during the first flood. In the absence of a clear scenario, we choose the simplest possible initial conditions : we assume that, initially, all tracers belong to the static layer : $\phi(x, t = 0) = 0$."

Similarly, the statement on line 33 that only a small proportion of tracers move during the entrainment regime needs justification. See Bradley and Tucker [2012] for example. In the first flood of that study, the proportion of mobile tracers was higher than in a subsequent, nearly identical flood.

The statement on line 33 is not about field studies. It describes the behavior of the plume predicted from equations (9) and (10), subject to the initial condition $\phi(x, t = 0) = 0$. We have modified the corresponding paragraph to clarify this point (page 6, line 16) :

> "With these initial conditions, the evolution of the plume follows two distinct regimes. At early times, the flow gradually dislodges tracers from the bed and entrains them in the bedload layer. During this entrainment regime, only a small proportion of the tracers move."

Page 7, line 24. It is misleading to say that most tracer studies are limited to a few hundred particles. Bradley and Tucker [2012] used nearly 900 tracers. Page 7. Line 25. The only way that statement that tracer concentration rapidly decreases to immeasurable levels could be correct is if no tracers were recovered. By definition, the recovery of even a single tracer particle is a measurable concentration.

These two statements (Page 7, line 24 and 25) are not meant as a criticism of field measurements. Instead, they question the use of the concentration for comparison between theory and field data. We have reworked the corresponding paragraph to clarify these two points (page 10, lines 17-24) :

> "Concentration, defined as the number of tracers per unit of area, depends on the area over which it is measured. Its value is meaningful when the measurement area is much larger than the distance between particles, and much smaller than the plume. During the entrainment regime, the plume develops a thin tail containing only a small proportion of tracers. Measuring the concentration profile during this regime is thus challenging. To our knowledge, only Sayre and Hubbell (1965) were able to measure consistent concentration profiles, using radioactive sand. In practice, most field campaigns involve a limited number of tracers (900 at most) (Liébault *et al.*, 2012; Bradley and Tucker, 2012; Phillips and Jerolmack, 2014; Bradley, 2017). It is thus more practical to consider integral quantities, such as the mean position of the plume $\langle x \rangle$, its variance σ^2, and its skewness γ."

C. Technical corrections

The word "pebbles" is misspelled as "peebles" in several locations (e.g. page 7, line 24)

Following the recommandation of reviewer #2, we have changed pebble into cobble.

Page 2, Line 10 should read "propagate downstream"

It now reads "travel downstream".

Page 2, Line 11 : The [Bradley and Tucker, 2012] reference is incorrectly cited as Nathan Bradley and Tucker.

We corrected the reference.

Page 9, line 22, should read "expand equations" and the reference to equation 31 is probably intended to be eq. 30.

We corrected the spelling of expand. Regarding the equations, (31) is the correct reference. But the development needs equation (30).

Page 15, line 20. This reference is incorrectly formatted.

We corrected the reference.

II. RESPONSE TO REFEREE 2

A. General comments

Summary : This manuscript develops an analytical model for the spreading of a plume of bed-load tracers. From the Erosion and Deposition model developed by Charru et al. (2004) they further develop analytical solutions for the mean, variance, and skewness of the spreading tracer plume. This model demonstrates and analytical solutions demonstrate that the spreading of bed-load tracers occupies two scaling regimes. The manuscript further demonstrates that the first three moments of the tracer plume can be set against each other to effectively remove the their dependence on time. They conclude with a useful description of how these results may be tested within a field setting.

General comments : Determining time in a river can be a somewhat abstract exercise and multiple authors have attempted it with varying degrees of success. This difficulty greatly impairs the utility of field tracers by requiring researchers to monitor both the hydrology and the sediment tracers themselves. However, this manuscript may have provided a framework that greatly increases the utility of field tracers. As the key insight of this manuscript results from setting the expressions for the mean, variance, and skewness against each other and effectively removing time from the problem. This is very clever and to my knowledge has not been done before despite its apparent simplicity (in many ways it would not have made any sense to compare these without the model and framework presented in this manuscript). This framework, if shown to be a reasonable predictor of natural rivers, could take tracers from something of a novelty measurement to a standard tool in bed load and mountain river monitoring campaigns.

I think that this manuscript does a good job of presenting the theory and model development, and I appreciate the authors discussion on how this result can be tested using tracer data as it is rare in the field of sediment transport that theory papers present easy to test hypotheses. These results will likely be of great interest to the bed load transport and mountain river scientific communities.

a *I have very few comments and they are related primarily to improving the clarity of several variable definitions. The manuscript would benefit from providing a physical description or picture of the flight length and flight duration. From Lajeunesse et al. (2010) these quantities represent the distance a particle travels from erosion to deposition and the duration of this movement, respectively. Those definitions are akin to the descriptions of 'steps' from the many papers that treat bed load probabilistically. In this manuscript though they seem to represent quantities that are much more akin to length and timescales that particles spend on the surface. Making this distinction very clear at the outset would help reader comprehension. Even if these quantities do not quite have an observed definition in the field it would help if the authors could expand on what they think they represent.*

As pointed out by the referee, the definition of the flight length and duration were ambiguous. We have modified the text following equation (5), to clarify these definitions in section 2 (page 4, line 17 - page 5, line 2) :

> "Laboratory experiments suggest that the deposition rate is proportional to the concentration of moving particles :
>
> $$D = \frac{n_{\mathrm{m}}}{\tau_f} \tag{3}$$
>
> where we introduce the average flight duration, $\tau_f = \ell_f / V$, and the average flight length, ℓ_f (Charru, Mouilleron, and Eiff, 2004; Lajeunesse, Malverti, and Charru, 2010). The flight length is the distance traveled by a mobile particle between its erosion and eventual deposition. Similarly, the flight duration is the time a particle spends in the bedload layer. In practice, measuring these quantities often proves difficult, since they depend on how one defines the mobile and the static layer (Lajeunesse *et al.*, 2017)."

In conclusion, I recommend that the manuscript be published in ESurf with a few very minor changes focused on enhancing the clarity.

[In the spirit of ESurf's open discussion period I have elected to read Reviewer 1's comments after the completion of my own review - I did not see anything within Reviewer 1's comments that should prevent this manuscript from being published, however the authors will need to provide greater clarification of their derivations to avoid the issues pointed out by reviewer 1.

A few comments on field tracers and what has been previously observed. To my knowledge all current field datasets report different relations for both the mean and variance scalings, but this is not surprising as these studies all use different metrics for time in a river (some variation of cumulative hydrologic forcing) and the fitted relations almost always stem from regression. Some of these regressions are physically justified, but the main point here is that a lot of different relations could be fit to the available datasets. That no one has really observed multiple mean and variance scaling regimes is not surprising. Without apriori knowledge of multiple scaling regimes and the locations of the break points it is unlikely that one would ever try to fit a complex function to these data due to the variability. With this current paper, there is no a reason to attempt more complicated models for the field data.

A final comment on the length of observation in field studies and a contribution that this manuscript makes. Even for the longest observed field studies (9 yrs as pointed out by Reviewer 1) it is not clear how long the rivers in those studies are actually 'on' (actively able to transport sediment). In a sense, a decade in a desert stream with few floods could be the same as a month in a tropical river that floods weekly. In terms of dynamics, maybe 9 yrs of data represents the entire scaling regime and maybe it still only represents the entrainment regime, because most of the time gravel rivers are effectively 'off'. This is key result of the current manuscript, as it provides a way to compare tracer studies by removing time, one of the more nebulous variables.

B. Specific comments

In several locations the term 'pebble' is used in place of what are likely cobbles. I understand what the authors mean, however more traditional geologists may find the use of the term confusing and misinterpret the size of the particles in question. I leave it up to the authors to choose.

Following the reviewer's advice, we have replace the term 'pebble' with 'cobble' everywhere in the mansucript.

Description of equation 1 - It is not immediately clear what the unit surface is? Is this the projected area of a grain (D^2) or the measurement window?

We are not referring to some specific surface, but to a unit of area, i.e. 1 m^2. E is the number of bed particles set in motion per unit of time and area. We have edited the text to clarify this ambiguity (page 3 - lines 26-27).

P. 4 Ln. 18 - The introduction of the 'flight length' should include a definition. Although it is defined in the cited papers, a short definition would benefit the readers comprehension of the concept. Something like the flight length represents the distance a particle travels from erosion to deposition.
P. 6 Ln. 16 - It now becomes clear to me that I am not sure exactly what tau_f *(the flight time) refers to physically. Is it the time of an individual flight (from erosion to deposition in the surface layer, on the order of seconds) or does it refer to a longer timescale that represents the time that the particle remains in a more mobile state?*

As discussed above, we have edited the text to clarify the definitions of the flight length and duration (page 4, line 17 - page 5, line 2).

P. 7 Ln. 24 & P. 11 Ln. 1 - 'peebles' likely a typo for pebbles. Though I would suggest cobbles per the earlier comment.

Bradley and Tucker (2012) or Bradley (2017) would be worth citing here as it represents the largest deployment to date.

As discussed above, we have substituded 'pebble' for 'cobble' everywhere in the manuscript. Following referee # 1, the first paragraph of section 5 has been reworked. It now includes citations of Bradley and Tucker (2012) and Bradley (2017).

P. 7 Ln. 26 - is 'the size' supposed to be the standard deviation ?

Indeed, the size is the standard deviation.To clarify this ambiguity, we have changed the text into " It is thus more practical to consider integral quantities, such as the mean position of the plume $\langle x \rangle$, its variance σ^2, and its skewness γ." (page 10, line 23).

P. 8 Ln. 10 - 'this' should be 'these' if the conditions are indeed plural.

Actually, there is only one condition : $\langle \delta \rangle$: $\langle \delta \rangle (t = 0) = \alpha + 1$.

P. 11 Ln. 6 - The preceding lines set up the notion that tracers maintain their conditions between floods (they don't move and in a sense are frozen), but this line suggests that this also applies to the actual floods. It is just a little confusing, during the flood isn't that when tracers might be mobile and thus changing their conditions ? Please clarify what is meant in this line and if floods should be included.

We removed this sentence, which was both confusing and unnecessary.

Pg. 11 Ln. 10-12 - Based on Paola et al. (1992), Phillips et al. (2013) have partially validated that the hydrograph intermittency is proportional to this same quantity. You might cite them here as a validation for the frameworks potentially broad applicability.

We have added a reference to Phillips, Martin, and Jerolmack (2013).

Pg. 12 Ln. 7 - It is not immediately clear to me what this line is saying. Could you reword this sentence to clarify its meaning. What I gathered from it is that the plume of tracers will remain in the entrainment scaling regime so long as the size (variance or range ?) is less than the length (mean ?) position. Is this what is meant ?

This sentence was not only unclear, it was also incorrect. We changed it to (page 16, lines 6) : "The entrainment regime lasts until the plume has traveled over a distance comparable to its initial size, that is until $\langle x \rangle - \langle x \rangle_0 \sim \sigma_0$."

RÉFÉRENCES

Agudo, J. and Wierschem, A., "Incipient motion of a single particle on regular substrates in laminar shear flow," Physics of Fluids **24**, 093302 (2012).

Bradley, D. N., "Direct observation of heavy-tailed storage times of bed load tracer particles causing anomalous superdiffusion," Geophysical Research Letters **44** (2017).

Bradley, D. N. and Tucker, G. E., "Measuring gravel transport and dispersion in a mountain river using passive radio tracers," Earth Surface Processes and Landforms **37**, 1034–1045 (2012).

Bradley, D. N., Tucker, G. E., and Benson, D. A., "Fractional dispersion in a sand bed river," Journal of Geophysical Research **115**, F00A09 (2010).

Charru, F., Mouilleron, H., and Eiff, O., "Erosion and deposition of particles on a bed sheared by a viscous flow," Journal of Fluid Mech.

519, 55–80 (2004).

Fathel, S., Furbish, D., and Schmeeckle, M., "Parsing anomalous versus normal diffusive behavior of bed load sediment particles," Earth Surface Processes and Landforms (2016).

Ganti, V., Meerschaert, M. M., Foufoula-Georgiou, E., Viparelli, E., and Parker, G., "Normal and anomalous diffusion of gravel tracer particles in rivers," Journal of Geophysical Research : Earth Surface **115** (2010).

Lajeunesse, E., Devauchelle, O., Lachaussée, F., and Claudin, P., "Bedload transport in laboratory rivers : The erosion–deposition model," (John Wiley & Sons Ltd., 2017) Chap. 15, pp. 415–438.

Lajeunesse, E., Malverti, L., and Charru, F., "Bedload transport in turbulent flow at the grain scale : experiments and modeling," J. Geophys. Res **115**, F04001, doi :10.1029/2009JF001628 (2010).

Liébault, F., Bellot, H., Chapuis, M., Klotz, S., and Deschâtres, M., "Bedload tracing in a high-sediment-load mountain stream," Earth Surface Processes and Landforms **37**, 385–399 (2012).

Martin, R. L., Jerolmack, D. J., and Schumer, R., "The physical basis for anomalous diffusion in bed load transport," Journal of Geophysical Research **117**, F01018 (2012).

Nikora, V., Habersack, H., Huber, T., and McEwan, I., "On bed particle diffusion in gravel bed flows under weak bed load transport," Water Resources Research **38**, 17–1 (2002).

Phillips, C. and Jerolmack, D., "Dynamics and mechanics of bed-load tracer particles," Earth Surf. Dynam. **2**, 513–530 (2014).

Phillips, C. B., Martin, R. L., and Jerolmack, D. J., "Impulse framework for unsteady flows reveals superdiffusive bed load transport," Geophysical Research Letters **40**, 1328–1333 (2013).

Sayre, W. and Hubbell, D., "Transport and dispersion of labeled bed material, north loup river, nebraska," Tech. Rep. 433-C (U.S. Geol. Surv. Prof. Pap., 1965).

Schumer, R., Meerschaert, M. M., and Baeumer, B., "Fractional advection-dispersion equations for modeling transport at the earth surface," Journal of Geophysical Research : Earth Surface **114** (2009).

Turowski, J. M., Badoux, A., and Rickenmann, D., "Start and end of bedload transport in gravel-bed streams," Geophysical Research Letters **38** (2011).

Weeks, E. R. and Swinney, H. L., "Anomalous diffusion resulting from strongly asymmetric random walks," Physical Review E **57**, 4915 (1998).

Zhang, Y., Meerschaert, M. M., and Packman, A. I., "Linking fluvial bed sediment transport across scales," Geophysical Research Letters **39** (2012).

Advection and dispersion of bedload tracers

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Abstract

We use the erosion-deposition model introduced by Charru et al. (2004) to simulate numerically the evolution of a plume of bedload tracers entrained by a steady flow. In this model, the propagation of the plume results from the stochastic exchange of particles between the bed and the bedload layer. We find a transition between two asymptotic regimes. At early time, the tracers, initially at rest, are gradually set into motion by the flow. During this entrainment regime, the plume, strongly skewed in the direction of propagation, continuously accelerates while spreading non-linearly. With time, the skewness of the plume eventually reaches a maximum value before decreasing. This marks the transition to an advection-diffusion regime in which the plume becomes increasingly symmetrical, spreads linearly, and advances at constant velocity. We derive analytically the expressions of the position, the variance and the skewness of the plume, and investigate their asymptotic regimes. Our model assumes steady state. In the field, however, bedload transport is intermittent. We show that the asymptotic regimes become insensitive to this intermittency when expressed in terms of the distance traveled by the plume. If this finding applies to the field, it might provide an estimate for the average bedload transport rate.

1 Introduction

Alluvial rivers transport the sediment that makes up their bed. From a mechanical standpoint, the flow of water applies a shear stress on the sediment particles, and entrains some of them downstream. When the shear stress is weak, the particles remain close to the bed surface as they travel (Shields, 1936). They roll, slide and bounce over the rough bed, until they settle down (Fernandez-Luque and Van Beek, 1976; Van Rijn, 1984; Nino and Garcia, 1994). This process is called bedload transport.

Bedload transport is inherently random (Einstein, 1937). A turbulent burst, or a collision with an entrained grain sometime dislodges a resting particle. The likeliness of this event depends on the specific arrangement of the surrounding particles. On average, however, the probability of entrainment is a function of macroscopic quantities such as shear stress and grain size (Ancey et al., 2008). Once dislodged, the velocity of a particle fluctuates significantly around its average (Lajeunesse et al., 2010a; Furbish et al., 2012b, c, a; Roseberry et al., 2012). Finally, the particle's return to rest is yet another random event. Overall, a bedload particle spends only a small fraction of its time in motion.

Altogether, the combination of these stochastic processes results in a downstream flux of particles. Fluvial geomorphologists measure this flux by collecting moving particles in traps or Helley-Smith samplers (Leopold and Emmett, 1976; Helley and Smith, 1971). The instantaneous sediment discharge fluctuates due to the inherent randomness of bedload transport. However, averaging measurements over time yields a consistent sediment flux (Liu et al., 2008).

An alternative approach to sediment-flux measurements is to follow the fate of tracer particles. In November 1960, Sayre and Hubbell (1965) deposited 18 kg of radioactive sand in the North-Loup river, a sand-bed stream located in Nebraska (USA). Using a scintillator detector, they observed that the plume of radioactive sand gradually spread as it was entrained downstream. Tracking cobbles in gravel-bed rivers reveals a similar behavior: tracers disperse as they travel downstream (Bradley et al., 2010; Bradley and Tucker, 2012; Hassan et al., 2013; Phillips et al., 2013).

The dispersion of the tracers, expressed as the variance of their location, results from the randomness of bedload transport. Nikora et al. (2002) identify three regimes with distinct time scales. A particle entrained by the flow repeatedly collides with the bed (Lajeunesse et al., 2017). At short time, between two collisions, particles move with the flow, and the variance increases as the square of time (Martin et al., 2012; Fathel et al., 2016). This regime is analogous to the ballistic regime of Brownian motion (Zhang et al., 2012; Fathel et al., 2016).

As the particle continues its course, collisions deviate its trajectory. In this intermediate regime, the variance increases non-linearly with time (Martin et al., 2012). Nikora et al. (2002) attribute this behavior to anomalous super-diffusion; but Fathel et al. (2016) contest their interpretation.

With time, tracers settle back on the bed, where they can remain trapped for a long time. How the distribution of resting times influences the long-term dispersion of tracers remains unknown. The data collected by Sayre and Hubbell (1965) are consistent with the existence of a diffusive regime, in which the variance increases linearly (Zhang et al., 2012). Other investigators, however, report either subdiffusion or super-diffusion (Nikora et al., 2002; Bradley, 2017). These anomalous diffusion regimes are sometimes modeled with fractional advection-dispersion equations (Schumer et al., 2009; Ganti et al., 2010; Bradley et al., 2010).

The variability of the stream discharge further complicates the interpretation of field data. Bedload transport occurs when the shear stress exceeds a threshold set by the grain size. Most rivers fulfill this condition only a small fraction of the time, making sediment transport highly intermittent (Phillips et al., 2013; Phillips and Jerolmack, 2014). The rate at which tracers spread thus depends not only on the inherent randomness of bedload transport, but also on the probability distribution of the river discharge (Ganti et al., 2010; Phillips et al., 2013; Bradley, 2017).

Laboratory experiments under well-controlled conditions isolate these two effects. For instance, Lajeunesse et al. (2017) tracked a plume of dyed particles in an experimental channel. Although the flow was constant in this experiment, the tracers still dispersed as they traveled downstream. In this case, dispersion resulted from the inherent randomness of bedload transport only. We can decompose this randomness into two components. First, the velocity fluctuations disperse the particles (Furbish et al., 2012a, c, 2016). Secondly, the random exchange of particles between the bedload layer, where particles travel, and the sediment bed, where particles are at rest, further disperses the particles (Lajeunesse et al., 2013, 2017). This effective diffusion also occurs in chromatography experiments, where a bonded phase exchanges the analyte with the flow (Van Genuchten and Wierenga, 1976).

In a recent paper, Lajeunesse et al. (2013) used the erosion-deposition model introduced by Charru et al. (2004) to derive the equations governing the evolution of a plume of tracers. Neglecting velocity fluctuations, they found that the second dispersion process, namely the exchange of particles between the bedload layer and the sediment bed, efficiently disperses the tracers. They also observed the transition between an initial transient and classical advection-diffusion. In the present paper, we further this investigation. Our objective is to derive formally the contribution of the advection-exchange of particles to the dispersion of a plume of tracers. To do so, we briefly rederive the equations governing the evolution of a plume of tracers (Sect. 2). We simulate numerically the propagation of a plume of tracers and discuss the nature of the two asymptotic regimes evidenced in Lajeunesse et al. (2013) (Sect. 3). We analyse the long-time advection-diffusion behavior of the plume and provide an analytical expression for the diffusion coefficient and the plume velocity (Sect. 4). We derive analytically the mean, the variance and the skewness of the tracers distribution and describe their asymptotic behavior in each regime (Sect. 5). Finally, we discuss the applicability of these results to the field (Sect. 6).

2 Entrainment of tracers

In most rivers, sediment are broadly distributed in size. This likely influences the dispersion of bedload tracers (Martin et al., 2012; Houssais and Lajeunesse, 2012; Pelosi et al., 2014). For the sake of simplicity, however, we restrict our analysis to a bed of uniform particles of size d_s. The bed is sheared by a flow, which applies a shear stress strong enough to entrain some particles. The latter remain confined in a thin bedload layer.

For moderate values of the shear stress, the concentration of moving sediments is small, and we can neglect the interactions between particles. The erosion-deposition model introduced by Charru et al. (2004) provides an accurate description of this dilute regime, in which bedload transport is controlled by the exchange of particles between the sediment bed and the bedload layer. This exchange sets the surface concentration of moving particles, n_{m}, through mass balance:

$$\frac{\partial n_{\mathrm{m}}}{\partial t} + V\,\frac{\partial n_{\mathrm{m}}}{\partial x} = E - D, \tag{1}$$

where we introduce the average particle velocity V. E is the erosion rate, defined as the number of bed particles set in motion per unit of time and area. Similarly, the deposition rate D is defined as the number of bedload particles settling on the bed per unit of time and area (Charru et al., 2004; Charru, 2006; Lajeunesse et al., 2010b; Seizilles et al., 2014; Lajeunesse et al., 2017).

To investigate the dispersion of bedload particles, we consider that some of them are marked (Fig. 1). We refer to these marked particles as “tracers”, and assume that their physical properties are the same as those of unmarked particles. With these assumptions, the mass balance for the tracers in the bedload layer reads

$$n_{\mathrm{m}}\,\frac{\partial \phi}{\partial t} + n_{\mathrm{m}}\,V\,\frac{\partial \phi}{\partial x} = E\,\psi - D\,\phi\,, \tag{2}$$

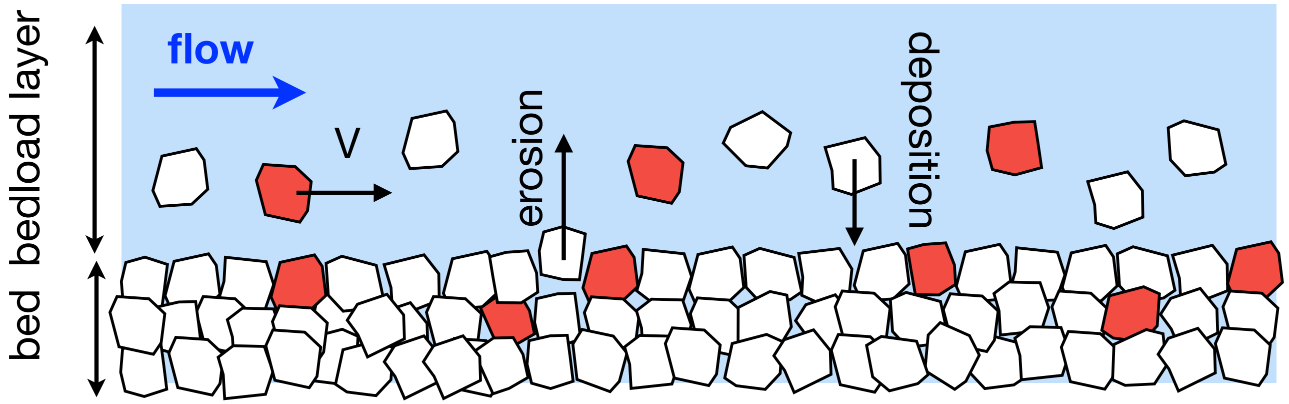

Figure 1. Granular bed sheared by a steady and uniform flow. The bed is a mixture of marked (red) and unmarked (white) grains.

where we introduce the proportion of tracers in the moving layer, ϕ. Similarly, ψ is the proportion of tracers on the bed surface.

When subjected to varying flow and sediment discharges, the bed of a stream accumulates or releases sediments (Gintz et al., 1996; Blom and Parker, 2004). Some particles may then be temporary buried within the bed, inducing streamwise dispersion (Crickmore and Lean, 1962; Pelosi et al., 2014). Here, we neglect this mechanism and restrict our analysis to steady and uniform sediment transport. Accordingly, we assume that erosion and deposition affects the bed over a depth of about one grain diameter only. This hypothesis holds if the departure from the entrainment threshold is small enough. With these assumptions, the mass balance for the tracers on the bed surface reads

$$n_{\mathrm{s}} \frac{\partial \psi}{\partial t} = D\,\phi - E\,\psi \tag{3}$$

where n_s is the surface concentration of particles at rest on the bed surface. Each of them occupies an area of about d_s^2. The surface concentration of particles at rest is therefore $n_s \sim 1/d_s^2$.

For steady and uniform transport, the surface concentration of moving particles, n, is constant. In addition, erosion and deposition balance each other:

$$E = D. \tag{4}$$

Laboratory experiments suggest that the deposition rate is proportional to the concentration of moving particles:

$$D = \frac{n_{\mathrm{m}}}{\tau_f} \tag{5}$$

where we introduce the average flight duration, $\tau_f = \ell_f / V$, and the average flight length, ℓ_f (Charru et al., 2004; Lajeunesse et al., 2010b). The flight length is the distance traveled by a mobile particle between its erosion and eventual deposition.

Similarly, the flight duration is the time a particle spends in the bedload layer. In practice, measuring these quantities often proves difficult, since they depend on how one defines the mobile and the static layer (Lajeunesse et al., 2017).

Combining equations (2), (3), (4) and (5) provides the set of equations that describe the propagation of the plume:

$$\frac{\partial \phi}{\partial t} + V\,\frac{\partial \phi}{\partial x} = \frac{1}{\tau_f}\,(\psi - \phi). \tag{6}$$

$$\frac{\partial \psi}{\partial t} = -\frac{\alpha}{\tau_f}\,(\psi - \phi) \tag{7}$$

where we define $\alpha = n_{\mathrm{m}}/n_{\mathrm{s}} \sim n_{\mathrm{m}} d_s^2$, the ratio of the concentration of moving particles to the concentration of static particles. This ratio is smaller than one. It is proportional to the intensity q_s of bedload transport:

$$\alpha \sim \frac{d_s^2}{V}\,q_s. \tag{8}$$

Complemented with initial and boundary conditions, equations (6) and (7) describe the evolution of the plume. In dimensionless form, they read

$$\frac{\partial \phi}{\partial \hat{t}} + \frac{\partial \phi}{\partial \hat{x}} = \psi - \phi \tag{9}$$

$$\frac{\partial \psi}{\partial \hat{t}} = -\alpha\,(\psi - \phi). \tag{10}$$

where $\hat{t} = t/\tau_f$ and $\hat{x} = x/\ell_f$ are dimensionless variables. For ease of notation, we drop the hat symbol in what follows.

A single parameter controls equations (9) and (10): the ratio of surface densities α, which characterizes the average distance between grains in the bedload layer. Since the erosion-deposition model assumes independent particles, we can only expect it to be valid when moving particles are sufficiently far away from each other, that is when α is small or, equivalently, when the Shields parameter is near threshold.

In the next section, we solve numerically equations (9) and (10).

3 Propagation of a plume of tracers

Laboratory measurements of bedload often use top-view images (Martin et al., 2012; Lajeunesse et al., 2017). Unless individual particles can be tracked, the tracers at rest are usually indistinguishable from those entrained by the flow. Separating the proportion of tracers in the moving layer, ϕ, from that on the bed surface, ψ, is practically impossible. Instead, top-view pictures show the total concentration of tracers:

$$c = \frac{n_m \phi + n_s \psi}{n_m + n_s} = \frac{\alpha}{\alpha+1}\phi + \frac{1}{\alpha+1}\psi. \tag{11}$$

Tracking sediment in rivers poses a similar problem. In general, one records the position of the tracers when the river stage is below the threshold of grain entrainment (Phillips et al., 2013; Phillips and Jerolmack, 2014). At the time of measurement, all tracers are therefore at rest. As a result, the proportion of mobile tracers vanishes ($\phi = 0$), and the total concentration of tracers reads $c = \psi/(\alpha + 1)$.

In summary, the proportions of mobile and static tracers, ϕ and ψ, naturally derive from mass balance (2) and (3). However their measurement proves difficult during active transport. On the other hand, experimental and field investigations provide the total concentration of tracers, c (Sayre and Hubbell, 1965; Lajeunesse et al., 2017). This quantity is conservative, as the total amount of tracers, $M = \int c\,\mathrm{d}x$, is preserved. In the following, we therefore focus on the concentration of tracers, c.

To study the evolution of the tracer concentration, we solve equations (9) and (10) numerically, using a finite volume scheme. We then compute the tracer concentration using equation (11) (Fig. 2).

The early evolution of the plume depends on initial conditions. In most field experiments, tracers are deposited at the surface of the river bed when the flow stage is low and sediment are motionless (Phillips et al., 2013). During floods, the river discharge increases and the shear stress eventually exceeds the entrainment threshold, setting in motion some of the grains. The entrainment of particles strongly depends on the arrangement of the bed: grains highly exposed to the flow move first (Charru et al., 2004; Turowski et al., 2011; Agudo and Wierschem, 2012). Several authors find that the tracers they disposed on the bed are more mobile during the first flood than during later ones (Bradley and Tucker, 2012). During the later floods, tracers gradually get trapped in the bed, and their average mobility decreases. On the other hand, Phillips and Jerolmack (2014) find no special mobility during the first flood. In the absence of a clear scenario, we choose the simplest possible initial conditions: we assume that, initially, all tracers belong to the static layer: $\phi(x,t=0) = 0$.

With these initial conditions, the evolution of the plume follows two distinct regimes. At early times, the flow gradually dislodges tracers from the bed and entrains them in the bedload layer. During this entrainment regime, only a small proportion of the tracers move. Consequently, the plume develops a thin tail in the downstream direction (Fig. 2a). The corresponding distribution of travel distances is strongly skewed towards the direction of propagation, a feature commonly observed in field experiments (Liébault et al., 2012; Phillips and Jerolmack, 2014).

With time, the plume moves downstream and spreads both upstream and downstream. As a result, the concentration rapidly decreases to small levels. The plume becomes gradually symmetrical and tends asymptotically towards a Gaussian distribution (Fig. 2b). This regime is reminiscent of classical diffusion.

To better illustrate this evolution, we introduce the mean position of the plume of tracers:

$$\langle x \rangle = \frac{1}{M} \int_{-\infty}^{\infty} c\,x\,\mathrm{d}x\,. \tag{12}$$

We also characterize its size with the variance:

$$\sigma^2 = \frac{1}{M} \int_{-\infty}^{\infty} c\,(x - \langle x \rangle)^2\,\mathrm{d}x \tag{13}$$

and its symmetry with the skewness:

$$\gamma = \frac{1}{M} \int_{-\infty}^{\infty} c\left(\frac{x - \langle x \rangle}{\sigma}\right)^3 \mathrm{d}x. \tag{14}$$

The evolution of these three moments is consistent with the existence of two asymptotic regimes (Fig. 3). At short time, the plume grows a thin tail downstream. This deformation causes the plume's skewness to increase as t^4. During this regime, the

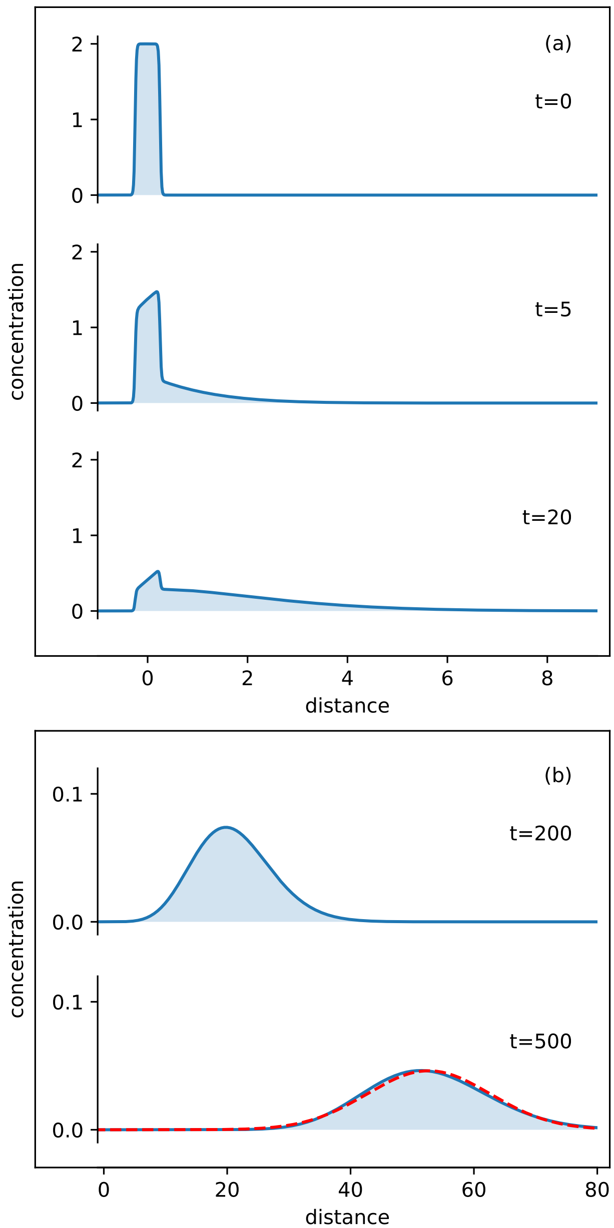

Figure 2. Evolution of the tracer concentration ($\alpha = 0.1$) obtained by solving numerically equations (9) and (10). (a) Early entrainment regime. (b) Relaxation towards the diffusive regime. Tracers are initially at rest, forming a symmetric plume of length $L = 0.5$ and mass $M = 1$. The concentration profile asymptotically tends towards a Gaussian distribution (dotted red line).

average location of the plume increases as t^2 and its variance grows as t^3. Although the variance increases non-linearly with time, the exponent, 3, is too large for super-diffusion (Weeks and Swinney, 1998).

After a characteristic time of the order of $\tau \approx \tau_f$, the skewness of the plume reaches a maximum (Fig. 3c). This corresponds to a drastic change of dynamics: the skewness starts decreasing as the plume becomes gradually more symmetrical. At long time, the plume of tracers advances at constant velocity and diffuses linearly with time (Fig. 3a and b). This regime, regardless of the value of α, corresponds to classical advection-diffusion.

Next, we establish the equivalence between diffusion and the long-time behavior of the tracers.

4 Advection-diffusion at long time

The diffusion at work in equations (9) and (10) results from the continuous exchange of particles between the bedload layer, where particles travel at the constant velocity V, and the sediment bed, where particles are at rest. The velocity difference between the two layers gradually smears out the plume and spreads it in the flow direction. This process occurs in a variety of physical systems in which layers moving at different velocities exchange a passive tracer. A typical example is Taylor dispersion, where a passive tracer diffuses across a Poiseuille flow in a circular pipe (Taylor, 1953). The combination of shear rate and transverse molecular diffusion generates an effective diffusion in the flow direction. Other exemples of effective diffusion include solute transport in porous media and chromatography (Van Genuchten and Wierenga, 1976).

To establish formally the equivalence between diffusion and the long-time behavior of the plume, we follow a reasoning similar to the one developed for chromatography (James et al., 2000). Equations (9) and (10) are equivalent to:

$$\frac{\partial c}{\partial t} + \frac{\alpha}{\alpha+1}\frac{\partial c}{\partial x} = \frac{\alpha}{(\alpha+1)^2}\frac{\partial \delta}{\partial x}, \tag{15}$$

$$\frac{\partial \delta}{\partial t} + \frac{1}{\alpha+1}\frac{\partial \delta}{\partial x} + (\alpha+1)\delta = \frac{\partial c}{\partial x}, \tag{16}$$

where we introduce $\delta = \psi - \phi$, the difference between the proportion of tracers on the sediment bed and that in the bedload layer. Eventually, these proportions equilibrate each other. At long time, we therefore expect the solution of equations (15) and (16) to relax towards steady state, for which δ is of order $\epsilon \ll 1$. Accordingly, we rewrite these two equations as

$$\frac{\partial c}{\partial t} + \frac{\alpha}{\alpha+1}\frac{\partial c}{\partial x} = \epsilon\,\frac{\alpha}{(\alpha+1)^2}\frac{\partial \delta}{\partial x}, \tag{17}$$

$$\frac{\partial \delta}{\partial t} + \frac{1}{\alpha+1}\frac{\partial \delta}{\partial x} + (\alpha+1)\,\delta = \frac{1}{\epsilon}\frac{\partial c}{\partial x}. \tag{18}$$

Introducing $T = \epsilon\, t$ and $X = \epsilon\, x$, and developing c and δ with respect to ϵ yields

$$\frac{\partial c_0}{\partial T} + \frac{\alpha}{\alpha+1}\frac{\partial c_0}{\partial X} = 0 \tag{19}$$

$$(\alpha+1)\,\delta_0 = \frac{\partial c_0}{\partial X} \tag{20}$$

at zeroth order, and

$$\frac{\partial c_1}{\partial T} + \frac{\alpha}{\alpha+1}\frac{\partial c_1}{\partial X} = \frac{\alpha}{(\alpha+1)^2}\frac{\partial \delta_0}{\partial X} \tag{21}$$

at first order.

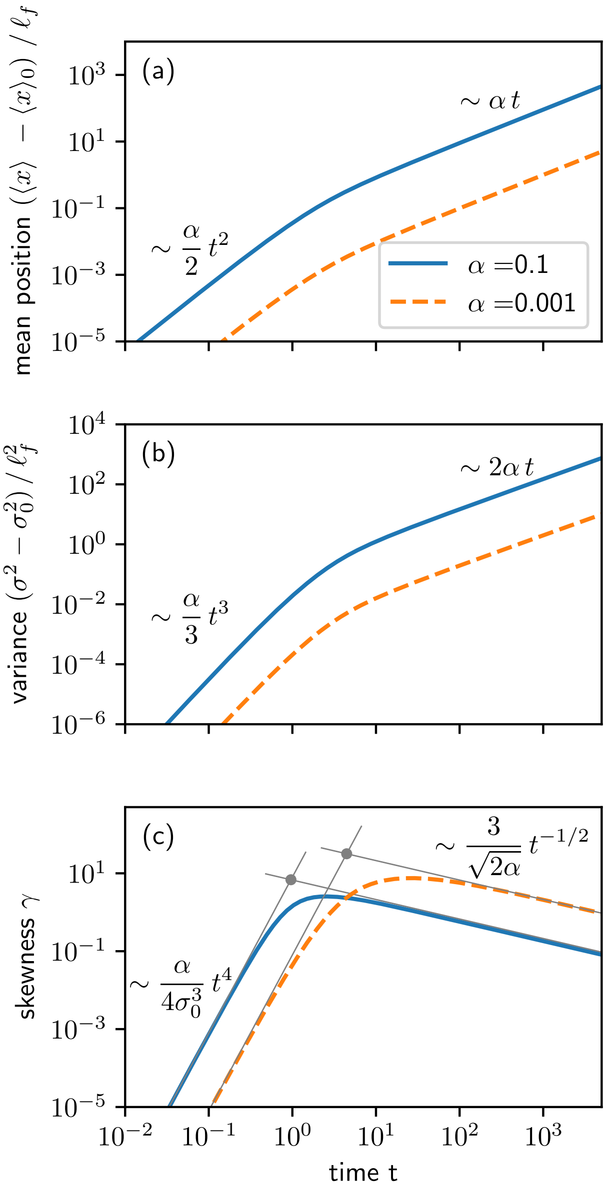

Figure 3. (a) Position, (b) variance and (c) skewness of a plume of tracers as a function of time for $\alpha = 0.1$ and $\alpha = 0.001$. We compute the evolution of these three quantities using equations (28), (33) and (38). The results agree exactly with numerical simulations. The asymptotic regimes of the skewness are represented with grey lines. Their intersection provides an estimate of the duration of the entrainment regime (see equation (45)).

Multiplying equation (21) by ϵ and summing the result with equation (19), we finally get

$$\frac{\partial c}{\partial t} + \frac{\alpha}{\alpha+1}\,\frac{\partial c}{\partial x} = \frac{\alpha}{(\alpha+1)^3}\,\frac{\partial^2 c}{\partial x^2}\,. \tag{22}$$

At long time, the transport of the tracers follows the advection-diffusion equation (22). We identify the advection velocity, U, which reads:

$$U = \frac{\alpha}{\alpha+1}\,\frac{\ell_f}{\tau_f} \sim \alpha\,\,\frac{\ell_f}{\tau_f}, \tag{23}$$

Likewise, the diffusion coefficient reads:

$$C_d = \frac{\alpha}{(\alpha+1)^3}\,\frac{\ell_f^2}{\tau_f} \sim \alpha\,\frac{\ell_f^2}{\tau_f}. \tag{24}$$

This asymptotic equivalence explains the advection-diffusion regime (Figures 2 and 3).

We interpret this formal derivation as follows. In the reference frame of the plume, a tracer at rest on the bed moves backward, while a tracer entrained in the bedload layer moves forward. At long time, the proportions of tracers in each layer equilibrate. Consequently, the probability that a tracer be entrained and move forward, equals that of deposition. In the reference frame of the plume, the exchange of particles between the bed and the bedload layer is thus a Brownian motion, hence the linear diffusion of the plume.

In the next section, we investigate the evolution of the location, the size and the symmetry of the plume as it propagates downstream.

5 Location, size and symmetry of the plume

Concentration, defined as the number of tracers per unit of area, depends on the area over which it is measured. Its value is meaningful when the measurement area is much larger than the distance between particles, and much smaller than the plume. During the entrainment regime, the plume develops a thin tail containing only a small proportion of tracers. Measuring the concentration profile during this regime is thus challenging. To our knowledge, only Sayre and Hubbell (1965) were able to measure consistent concentration profiles, using radioactive sand. In practice, most field campaigns involve a limited number of tracers (900 at most) (Liébault et al., 2012; Bradley and Tucker, 2012; Phillips and Jerolmack, 2014; Bradley, 2017). It is thus more practical to consider integral quantities, such as the mean position of the plume $\langle x\rangle$, its variance σ^2, and its skewness γ.

Multiplying equation (15) by x and integrating over space provides the evolution equation for the mean position:

$$\frac{\partial\langle x\rangle}{\partial t} = \frac{\alpha}{\alpha+1} - \frac{\alpha}{(\alpha+1)^2}\,\langle\delta\rangle \tag{25}$$

where

$$\langle\delta\rangle = \frac{1}{M}\int \delta\,dx \tag{26}$$

is the average difference between the proportion of tracers on the sediment bed and in the bedload layer. To solve equation (25), we need an equation for $\langle\delta\rangle$. The latter is obtained by integrating (16) over space:

$$\frac{\partial\langle\delta\rangle}{\partial t} = -(\alpha+1)\,\langle\delta\rangle \tag{27}$$

Equations (25) and (27) describe the downstream motion $\langle x\rangle$ of the plume. To solve them, we need to specify initial conditions. As discussed in section 3, we consider that all tracers initially belong to the static layer i.e. $\phi(x,t=0)=0$. This condition and the conservation of mass, $\langle c\rangle = 1$, provide initial conditions for $\langle\delta\rangle$: $\langle\delta\rangle(t=0)=\alpha+1$. With this condition, equations (25) and (27) integrate into

$$\langle\, x\rangle - \langle\, x\rangle_0 = \frac{\alpha}{\alpha+1}\,t + \frac{\alpha}{(\alpha+1)^2}\,\left(e^{-(\alpha+1)t}-1\right) \tag{28}$$

where $\langle\, x\rangle_0$ is the initial position of the plume.

We now focus on the variance of the plume. Multiplying (15) by x^2 and integrating over space yields the evolution equation for the second moment of the tracer distribution:

$$\frac{\partial\langle x^2\rangle}{\partial t} = \frac{2\alpha}{(\alpha+1)}\,\langle x\rangle - \frac{2\alpha}{(\alpha+1)^2}\langle x\delta\rangle \tag{29}$$

where

$$\langle x\,\delta\rangle = \frac{1}{M}\int x\,\delta\,dx. \tag{30}$$

is the first moment of δ. To solve equation (29), we need an equation for this intermediate quantity. We obtain it by multiplying (16) by x and integrating over space:

$$\frac{\partial\langle x\delta\rangle}{\partial t} = -1-(\alpha+1)\langle x\delta\rangle + \frac{\langle\delta\rangle}{\alpha+1} \tag{31}$$

At time $t=0$, $\langle x\delta\rangle(t=0)=(\alpha+1)\langle x\rangle_0$. Equations (29) and (31) with this initial condition provide the expression of the second moment of the tracer distribution:

$$\begin{aligned}\langle x^2\rangle = \langle x^2\rangle_0 + \frac{2\,\alpha}{(\alpha+1)^3}\left(t\,+\frac{2-\alpha}{\alpha+1}\right)e^{-(\alpha+1)\,t} + \frac{\alpha^2}{(\alpha+1)^2}\,t^2 \\ + \frac{2\alpha\,(1-\alpha)}{(\alpha+1)^3}\,t + \frac{2\alpha\,(\alpha-2)}{(\alpha+1)^4}\end{aligned} \tag{32}$$

where $\langle x^2\rangle_0$ is the initial value of the second moment of the tracer distribution. We then deduce the variance of the plume from:

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2. \tag{33}$$

We follow a similar procedure to derive the skewness of the plume. Multiplying (15) by x^3 and integrating over space yields the evolution equation for the third moment of the tracer distribution:

$$\frac{\partial\langle x^3\rangle}{\partial t} = \frac{3\alpha}{(\alpha+1)}\,\langle x^2\rangle - \frac{3\alpha}{(\alpha+1)^2}\langle x^2\delta\rangle \tag{34}$$

where

$$\langle x^2\,\delta\rangle = \frac{1}{M}\int x^2\,\delta\,dx. \tag{35}$$

is the second moment of δ. Multiplying (16) by x^2 and integrating over space provides the evolution equation for this intermediate quantity:

$$\frac{\partial\langle x^2\delta\rangle}{\partial t} = -(\alpha+1)\langle x^2\delta\rangle + \frac{2}{\alpha+1}\langle x\delta\rangle - 2\langle x\rangle \tag{36}$$

At time $t=0$, $\langle x^2\delta\rangle = (\alpha+1)\langle x^2\rangle_0$ and $\langle x^3\rangle = 0$. With these initial conditions, equations (34) and (36) provide the expression of $\langle x^3\rangle$:

$$\begin{aligned}\langle x^3\rangle = \frac{3\alpha}{\alpha+1}\left(\sigma_0^2 + \frac{2\alpha^2-8\alpha+2}{(\alpha+1)^4}\right) t \\ + \frac{3\alpha}{(\alpha+1)^2}\left(\sigma_0^2 + \frac{2\alpha^2-12\alpha+6}{(\alpha+1)^4}\right)\left(e^{-(\alpha+1)t}-1\right) \\ + \frac{3\alpha}{(\alpha+1)^4}\left(t - 4\frac{\alpha-1}{\alpha+1}\right) t\,e^{-(\alpha+1)t} \\ + \frac{\alpha^3}{(\alpha+1)^3}\left(t - \frac{3(\alpha-2)}{\alpha(\alpha+1)}\right) t^2\end{aligned} \tag{37}$$

from which we deduce the skewness of the plume:

$$\gamma = \frac{\langle x^3\rangle - 3\,\langle x\rangle\,\sigma^2 - \langle x\rangle^3}{\sigma^3} \tag{38}$$

Equations (28), (32), (33), (37), and (38) represent the evolution of the mean, the variance and the skewness of the tracers distribution, that is, its migration, spreading and symmetry. They do not require any assumption, other than the ones of the model itself, and agree exactly with numerical simulations (Fig. 3).

As discussed in section 3, numerical simulations reveal a transient during which the tracers, initially at rest, are gradually set into motion by the flow (Fig. 3). During this entrainment regime, the plume continuously accelerates, spreads non-linearly and becomes increasingly asymmetrical. To characterize this regime, we expand equations (28), (32), (33), (37), and (38) to leading order in time:

$$\langle x\rangle - \langle x\rangle_0 \sim \frac{\alpha}{2}\,t^2 \tag{39}$$

$$\sigma^2 - \sigma_0^2 \sim \frac{\alpha}{3}\,t^3 \tag{40}$$

$$\gamma \sim \frac{\alpha}{4\,\sigma_0^3}\,t^4. \tag{41}$$

These three equations are consistent with our numerical simulations (Fig. 3).

Anomalous diffusion arises from heavy-tailed distributions, of either the step length or the waiting time (Weeks and Swinney, 1998). The erosion-deposition model contains no such ingredient. Here the fast increase of the variance results from the exchange of particles between the sediment bed and the bedload layer, at the beginning of the experiment. Over a time shorter than the flight duration τ_f, the tracers entrained by the flow do not settle back on the bed. They form a thin tail, which leaves the main body of the plume, and moves downstream at the average particle velocity V (Fig. 2a). The plume therefore consists of a main body of virtually constant concentration, followed by a thin tail of length $\propto Vt$. Accordingly, we can split the integral that defines its mean position, equation (12), into two terms. The first one, obtained by integrating cx over the main body of the plume, yields the initial position of the plume $\langle x\rangle_0$. The second one, obtained by integrating cx over a tail of length Vt, scales as t^2. Summing these contributions yields equation (39). Similar reasonings yield equations (40) and (41) for the variance and the skewness.

With time, the plume enters the diffusive regime. Its velocity and its spreading rate relax towards constants while its skewness decreases (Fig. 3). We derive the corresponding asymptotic behavior by expanding equations (28), (32), (33), (37), and (38) at long time:

$$\langle x\rangle - \langle x\rangle_0 \sim \frac{\alpha}{\alpha+1}\, t \sim \alpha\, t \tag{42}$$

$$\sigma^2 - \sigma_0^2 \sim 2\frac{\alpha}{(\alpha+1)^3}\, t \sim 2\,\alpha\, t \tag{43}$$

$$\gamma \sim \frac{3}{\sqrt{2\,\alpha}}\,\frac{1}{\sqrt{t}} \tag{44}$$

The asymptotic regimes (42) and (43) are consistent with the expressions derived in section 4.

The transition between the entrainment and the diffusive regime occurs when the skewness reaches its maximum value. Equating the skewness estimated from (41) and (44) provides the approximate duration of the entrainment regime, τ. We find

$$\tau_e = (72)^{1/9}\left(\frac{\sigma_0^2}{\alpha}\right)^{1/3}\tau_f \tag{45}$$

which compares well with our numerical simulations (Fig. 3). The duration of the entrainment regime increases with the initial size of the plume and decreases with the intensity of sediment transport.

The asymptotic regimes (39), (40), (41), (42), (43) and (44) assume that sediment transport is in steady state. In the next section, we discuss the intermittency of bedload transport in natural streams.

6 Intermittency of bedload transport

Our description of the plume of tracers is based on the assumption that sediment transport is in steady state. This hypothesis is often satisfied in laboratory flumes (Lajeunesse et al., 2017). In a river, it may be met for up to a few days (Sayre and Hubbell, 1965). At longer time scales, however, most rivers alternate between low-flow stages during which sediment is immobile, and floods, during which bed particles are entrained downstream (Phillips and Jerolmack, 2016). Bedload transport is thus intermittent.

The intermittency of bedload transport influences the propagation of tracers in several ways. First of all, sediment transport during a flood modifies the structure of the bed (Lenzi et al., 2004; Turowski et al., 2009, 2011). As a result, the proportion of tracers in the bedload layer and in the bed, ϕ and ψ, likely change from one flood to the next. In a effort to address this

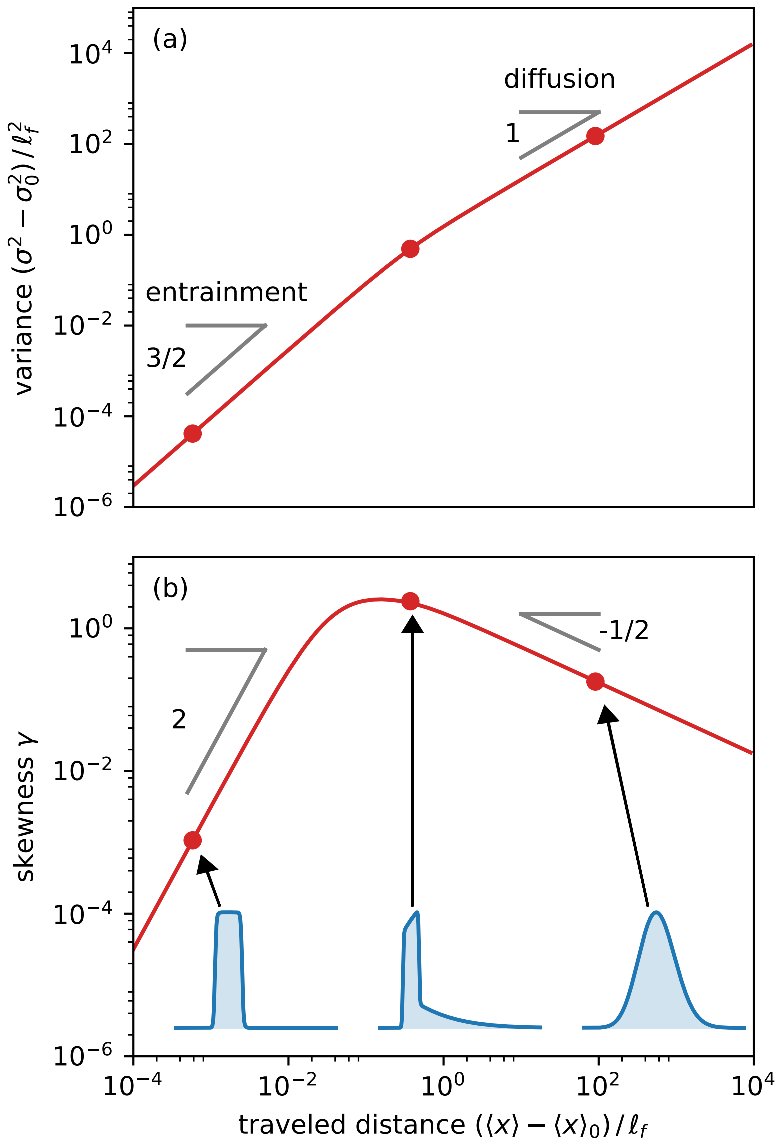

Figure 4. (a) Variance and (b) skewness of a plume of tracers as a function of traveled distance ($\alpha = 0.1$). These three quantities are calculated from equations (28), (33) and (38). Inset: concentration profiles (blue) illustrating the shape of the plume during the entrainment regime, at the transition between the entrainment and the diffusive regime and in the diffusive regime.

question, P. Allemand and collaborators recently implemented the survey of a river located in Basse-Terre Island (Guadeloupe archipelago). Their preliminary observations reveal that the cobbles deposited at the end of a flood are the first entrained at the beginning of the next (P. Allemand, personal communication, June 30, 2017). Based on this observation, we speculate that a tracer belonging to the bedload layer at the end of a flood will still be part of the bedload layer at the beginning of the next one. Similarly, a tracer locked in the bed at the end of a flood will belong to the static layer at the beginning of the next one. In other words, we assume that tracers freeze between two floods.

If this assumption holds, the simplest way to account for bedload intermittency is to assume that the river alternates between two representative stages: 1) a low-flow stage during which tracers are immobile; 2) a flood stage, characterized by a representative sediment flux $q_s \sim \alpha V/d_s^2$, during which tracers propagate downstream (Paola et al., 1992; Phillips et al., 2013). Following this model, we may extrapolate our results to the field, provided we rescale time with respect to an intermittency factor $I = T_e/T$, where T is the total duration of elapsed time, while T_e is the time during which sediments are effectively in motion (Paola et al., 1992; Parker et al., 1998; Phillips et al., 2013).

In practice, evaluating the intermittency factor requires continuous monitoring of the river discharge, and a correct estimate of the entrainment threshold. Liébault et al. (2012), for instance, monitored the location of tracer cobbles deposited in the Bouinenc stream (France) during 2 years. Over this period, the motion of the tracers resulted from 55 floods, for a total duration of 42 days. Sediments were thus in motion less than $I = 12\%$ of the time.

Here, we suggest another way to circumvent the intermittency of sediment transport. Plotting the plume variance, $(\sigma^2 - \sigma_0^2)$, and its skewness, γ, as a function of traveled distance, $\langle x\rangle - \langle x\rangle_0$, eliminates time from the equations (Fig. 4). In this plot, the position of the plume acts as a proxy for the effective duration of sediment transport, T_e. The resulting curves are thus filtered from transport intermittency (Fig. 4).

The entrainment regime corresponds to small traveled distances. In this regime, both the size of the plume and its asymmetry increase with traveled distance (Fig. 4). Equations (39), (40) and (41) describe the early evolution of the plume. Eliminating time by combining them, we find the behavior of the plume for short traveled distances:

$$\sigma^2 - \sigma_0^2 = \sqrt{\frac{8\,\ell_f}{9\,\alpha}}\,\left(\langle x\rangle - \langle x\rangle_0\right)^{3/2}, \tag{46}$$

$$\gamma = \frac{\ell_f}{\alpha\,\sigma_0^3}\,\left(\langle x\rangle - \langle x\rangle_0\right)^2. \tag{47}$$

As discussed in section 5, these scalings result from the gradual entrainment of the tracers that are initially trapped in the bed.

After the plume has traveled over a distance roughly equal to the flight length, its skewness reaches a maximum value and starts decreasing. This change of dynamics indicates the transition towards the diffusive regime. Equations (42), (43) and (44) provide the long-term behavior of the plume :

$$\sigma^2 - \sigma_0^2 \sim 2\,\ell_f\,\left(\langle x\rangle - \langle x\rangle_0\right), \tag{48}$$

$$\gamma = \frac{3}{\sqrt{2}}\sqrt{\frac{\ell_f}{\langle x\rangle - \langle x\rangle_0}}. \tag{49}$$

The linear increase of the variance with the distance traveled by the plume is the signature of standard diffusion (see section 5).

Equating the skewness estimated from (47) and (49) provides the position $\langle x\rangle_{max}$ at which the skewness reaches its maximum:

$$\langle x\rangle_{max} - \langle x\rangle_0 \sim \left(\frac{3\,\alpha}{\sqrt{2}}\right)^{2/5} \left(\frac{\sigma_0^6}{\ell_f}\right)^{1/5}. \tag{50}$$

The entrainment regime lasts until the plume has traveled over a distance comparable to its initial size, that is until $\langle x\rangle - \langle x\rangle_0 \sim \sigma_0$.

When expressed in terms of the distance traveled by the plume, the asymptotic regimes are insensitive to the intermittency of bedload transport. They are thus a robust test of our model, and can help us interpret field data. Let us assume that a dataset records the evolution of a plume of tracers released in a river, over a distance long enough to explore both the entrainment and the diffusive regime. During the diffusive regime, the skewness decreases with the traveled distance. A fit of the data with equation (49) yields the flight length, ℓ_f. Knowing the latter, we could use equation (47) to estimate the intensity of sediment transport, α, from the evolution of the skewness during the entrainment regime.

According to section 5, the skewness reaches a maximum after a time τ_e (equation (45)). Taking into account the intermittency of bedload transport in natural streams, we expect that this maximum is reached when

$$t = (72)^{1/9} \left(\frac{\sigma_0^2}{\alpha}\right)^{1/3} \frac{\tau_f}{I}, \tag{51}$$

where I is the intermittency factor. Identifying this maximum in a field experiment thus yields the ratio τ_f/I. Combining the latter with our estimates of the flight length, ℓ_f, and the intensity of sediment transport, α, should provide us with the average sediment transport rate in the river:

$$\bar{q_s} = I\,\alpha\,d_s^2\,\frac{l_f}{\tau_f}. \tag{52}$$

7 Conclusion

We used the erosion-deposition model introduced by Charru et al. (2004) to describe the evolution of a plume of bedload tracers entrained by a steady flow. In this model, the propagation of the plume results from the stochastic exchange of particles between the bed and the bedload layer. This mechanism is reminiscent of the propagation of tracers in a porous medium (Berkowitz and Scher, 1998). The evolution of the plume depends on two control parameters: its initial size, σ_0 , and the intensity of sediment transport, α.

Our model captures in a single theoretical framework the transition between two asymptotic regimes : 1) an early entrainment regime during which the plume spreads non-linearly, 2) a late-time relaxation towards classical advection-diffusion. The latter regime is consistent with previous observations (Nikora et al., 2002; Zhang et al., 2012).

When expressed in terms of the distance traveled by the plume, the asymptotic regimes are insensitive to the intermittency of bedload transport in natural streams. According to this model, it should be possible to estimate the particle flight length and the average bedload transport rate from the evolution of the variance and the skewness of a plume of tracers in a river.

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References

Agudo, J. and Wierschem, A.: Incipient motion of a single particle on regular substrates in laminar shear flow, Physics of Fluids, 24, 093 302, 2012.

Ancey, C., Davison, A., Bohm, T., Jodeau, M., and Frey, P.: Entrainment and motion of coarse particles in a shallow water stream down a steep slope, J. Fluid Mech., 595, 83–114, doi: 10.1017/S0022112007008 774, 2008.

Berkowitz, B. and Scher, H.: Theory of anomalous chemical transport in random fracture networks, Physical Review E, 57, 5858, 1998.

Blom, A. and Parker, G.: Vertical sorting and the morphodynamics of bed form–dominated rivers: A modeling framework, Journal of Geophysical Research: Earth Surface (2003–2012), 109, 2004.

Bradley, D. N.: Direct Observation of Heavy-Tailed Storage Times of Bed Load Tracer Particles Causing Anomalous Superdiffusion, Geophysical Research Letters, 44, 2017.

Bradley, D. N. and Tucker, G. E.: Measuring gravel transport and dispersion in a mountain river using passive radio tracers, Earth Surface Processes and Landforms, 37, 1034–1045, 2012.

Bradley, D. N., Tucker, G. E., and Benson, D. A.: Fractional dispersion in a sand bed river, Journal of Geophysical Research, 115, F00A09, 2010.

Charru, F.: Selection of the ripple length on a granular bed sheared by a liquid flow, Phys. Fluids, 18, 121 508–1–121 508–9, doi: 0.1017/S002211200500 786X, 2006.

Charru, F., Mouilleron, H., and Eiff, O.: Erosion and deposition of particles on a bed sheared by a viscous flow, Journal of Fluid Mech., 519, 55–80, 2004.

Crickmore, M. and Lean, G.: The measurement of sand transport by means of radioactive tracers, in: Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, vol. 266, pp. 402–421, The Royal Society, 1962.

Einstein, H. A.: Bed load transport as a probability problem, in: Sedimentation: 746 Symposium to Honor Professor H.A. Einstein, 1972. translation from 747 German of H.A. Einstein doctoral thesis. Originally presented to Federal Institute of Technology, Zurich, Switzerland, 1937, pp. C1 – C105, 1937.

Fathel, S., Furbish, D., and Schmeeckle, M.: Parsing anomalous versus normal diffusive behavior of bed load sediment particles, Earth Surface Processes and Landforms, 2016.

Fernandez-Luque, R. and Van Beek, R.: Erosion and transport of bed-load sediment, Journal of Hydraulic Research, 14, 127–144, 1976.

Furbish, D., Ball, A., and Schmeeckle, M.: A probabilistic description of the bed load sediment flux: 4. Fickian diffusion at low transport rates, Journal of Geophysical Research, 117, F03 034, 2012a.

Furbish, D., Haff, P., Roseberry, J., and Schmeeckle, M.: A probabilistic description of the bed load sediment flux: 1. Theory, Journal of Geophysical Research, 117, F03 031, 2012b.

Furbish, D., Roseberry, J., and Schmeeckle, M.: A probabilistic description of the bed load sediment flux: 3. The particle velocity distribution and the diffusive flux, Journal of Geophysical Research, 117, F03 033, 2012c.

Furbish, D. J., Fathel, S. L., Schmeeckle, M. W., Jerolmack, D. J., and Schumer, R.: The elements and richness of particle diffusion during sediment transport at small timescales, Earth Surface Processes and Landforms, 2016.

Ganti, V., Meerschaert, M. M., Foufoula-Georgiou, E., Viparelli, E., and Parker, G.: Normal and anomalous diffusion of gravel tracer particles in rivers, Journal of Geophysical Research: Earth Surface, 115, 2010.

Gintz, D., Hassan, M. A., and SCHMIDT, K.-H.: Frequency and magnitude of bedload transport in a mountain river, Earth Surface Processes and Landforms, 21, 433–445, 1996.

Hassan, M. A., Voepel, H., Schumer, R., Parker, G., and Fraccarollo, L.: Displacement characteristics of coarse fluvial bed sediment, Journal of Geophysical Research: Earth Surface, 118, 155–165, 2013.

Helley, E. J. and Smith, W.: Development and calibration of a pressure-difference bedload sampler, Geological Survey Open-File Report, December 3, 1971., 1971.

Houssais, M. and Lajeunesse, E.: Bedload transport of a bimodal sediment bed, Journal of Geophysical Research: Earth Surface, 117, 2012.

James, F., Postel, M., and Sepúlveda, M.: Numerical comparison between relaxation and nonlinear equilibrium models. Application to chemical engineering, Physica D: Nonlinear Phenomena, 138, 316–333, 2000.

Lajeunesse, E., Malverti, L., and Charru, F.: Bedload transport in turbulent flow at the grain scale: experiments and modeling, J. Geophys. Res. Earth Surface, 115, F04 001, doi:10.1029/2009JF001 628, 2010a.

Lajeunesse, E., Malverti, L., and Charru, F.: Bedload transport in turbulent flow at the grain scale: experiments and modeling, J. Geophys. Res, 115, F04 001, doi:10.1029/2009JF001 628, 2010b.

Lajeunesse, E., Devauchelle, O., Houssais, M., and Seizilles, G.: Tracer dispersion in bedload transport., Adv. Geosci., 37, 1–6, 2013.

Lajeunesse, E., Devauchelle, O., Lachaussée, F., and Claudin, P.: Bedload Transport in Laboratory Rivers: The Erosion–Deposition Model, chap. 15, pp. 415–438, John Wiley & Sons Ltd., 2017.

Lenzi, M., Mao, L., and Comiti, F.: Magnitude-frequency analysis of bed load data in an Alpine boulder bed stream, Water Resources Research, 40, 2004.

Leopold, L. B. and Emmett, W. W.: Bedload measurements, East Fork River, Wyoming, Proceedings of the National Academy of Sciences, 73, 1000–1004, 1976.

Liébault, F., Bellot, H., Chapuis, M., Klotz, S., and Deschâtres, M.: Bedload tracing in a high-sediment-load mountain stream, Earth Surface Processes and Landforms, 37, 385–399, 2012.

Liu, Y., Metivier, F., Lajeunesse, E., Lancien, P., Narteau, C., and Meunier, P.: Measuring bed load in gravel bed mountain rivers : averaging methods and sampling strategies, Geodynamica Acta, 21, 81–92, doi:10.3166/ga.21.81–92, 2008.

Martin, R. L., Jerolmack, D. J., and Schumer, R.: The physical basis for anomalous diffusion in bed load transport, Journal of Geophysical Research, 117, F01 018, 2012.

Nikora, V., Habersack, H., Huber, T., and McEwan, I.: On bed particle diffusion in gravel bed flows under weak bed load transport, Water Resources Research, 38, 17–1, 2002.

Nino, Y. and Garcia, M.: Gravel saltation. Part I: Experiments, Water Resources Research, 30, 1907–1914, 1994.

Paola, C., Hellert, P., and Angevine, C.: The large-scale dynamics of grain-size variation in alluvial basins, 1 : theory, Basin Research, 4, 73–90, 1992.

Parker, G., Paola, C., Whipple, K., Mohrig, D., Toro-Escobar, C., Halverson, M., and Skoglund, T.: Alluvial fans formed by channelized fluvial and sheet flow. II: Application, Journal of Hydraulic Engineering, 124, 996–1004, 1998.

Pelosi, A., Parker, G., Schumer, R., and Ma, H.-B.: Exner-Based Master Equation for transport and dispersion of river pebble tracers: Derivation, asymptotic forms, and quantification of nonlocal vertical dispersion, Journal of Geophysical Research: Earth Surface, 119, 1818–1832, 2014.

Phillips, C. and Jerolmack, D.: Dynamics and mechanics of bed-load tracer particles, Earth Surf. Dynam., 2, 513–530, 2014.

Phillips, C. B. and Jerolmack, D. J.: Self-organization of river channels as a critical filter on climate signals, Science, 352, 694–697, 2016.

Phillips, C. B., Martin, R. L., and Jerolmack, D. J.: Impulse framework for unsteady flows reveals superdiffusive bed load transport, Geophysical Research Letters, 40, 1328–1333, 2013.

Roseberry, J., Schmeeckle, M., and Furbish, D.: A probabilistic description of the bed load sediment flux: 2. Particle activity and motions, Journal of Geophysical Research, 117, F03 032, 2012.

Sayre, W. and Hubbell, D.: Transport and dispersion of labeled bed material, North Loup River, Nebraska, Tech. Rep. 433-C, U.S. Geol. Surv. Prof. Pap., 1965.

Schumer, R., Meerschaert, M. M., and Baeumer, B.: Fractional advection-dispersion equations for modeling transport at the Earth surface, Journal of Geophysical Research: Earth Surface, 114, 2009.

Seizilles, G., Lajeunesse, E., Devauchelle, O., and Bak, M.: Cross-stream diffusion in bedload transport, Phys. of Fluids, 26, 013 302, 2014.

Shields, A. S.: Anwendung der Aehnlichkeitsmechanik und der Turbulenzforschung auf die Geschiebebewegung, Mitteilung der Preussischen Versuchsanstalt fur Wasserbau und Schiffbau, 26, 1936.

Taylor, G.: Dispersion of soluble matter in solvent flowing slowly through a tube, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 219, 186–203, 1953.

Turowski, J. M., Yager, E. M., Badoux, A., Rickenmann, D., and Molnar, P.: The impact of exceptional events on erosion, bedload transport and channel stability in a step-pool channel, Earth Surface Processes and Landforms, 34, 1661–1673, 2009.

Turowski, J. M., Badoux, A., and Rickenmann, D.: Start and end of bedload transport in gravel-bed streams, Geophysical Research Letters, 38, 2011.

Van Genuchten, M. T. and Wierenga, P.: Mass transfer studies in sorbing porous media I. Analytical solutions, Soil Science Society of America Journal, 40, 473–480, 1976.

Van Rijn, L.: Sediment transport, part I: bed load transport, Journal of hydraulic Engineering, 110, 1431–1456, 1984.

Weeks, E. R. and Swinney, H. L.: Anomalous diffusion resulting from strongly asymmetric random walks, Physical Review E, 57, 4915, 1998.

Zhang, Y., Meerschaert, M. M., and Packman, A. I.: Linking fluvial bed sediment transport across scales, Geophysical Research Letters, 39, 2012.