Reply to two referees' comments on "Scaling and similarity of a stream-power incision and linear diffusion landscape evolution model" by Theodoratos et al.

We are grateful to referee Greg Tucker and the anonymous referee for their feedback on our manuscript. In the following response to their reviews, we first quote their comments (in indented blocks of italic text) and then respond (in normal text). In the quoted comments of the referees, the numbering of pages, lines, and sections refers to the submitted manuscript, and in our responses, the numbering follows the revised manuscript with marked-up changes.

Nikos Theodoratos, Hansjörg Seybold, and James Kirchner

1. Response to referee G.E. Tucker

> *Overview:*
>
> *This paper presents an insightful analysis of the simplest form of governing equation for drainage basin evolution. The authors present a dimensional analysis of the equation, showing that by treating horizonal and vertical scales separately, and by using the parameters themselves as the basis for normalization, one arrives at a simple and elegant dimensionless rendering—which contains important lessons about similarity. From the analysis emerge three characteristic scales, representing horizontal length, vertical distance (height), and time, respectively. And from this scale analysis emerge valuable insights into a number of different aspects of basin and landscape evolution, such as: what sets the scale of transition between hillslopes and valleys? How do the efficiencies of hillslope transport and channel incision influence rates of drainage divide migration? How are process parameters, which represent relative efficiencies of a geomorphic processes, manifested in quantitative terrain metrics? This is a theoretical exercise that shows very nicely how we can usefully think about the implications of the "stream power plus diffusion" model of landscape evolution: what are the key scales to consider, and how the parameters relate to one another. And although this ground has been covered before to some extent (most notably in the work of Garry Willgoose and colleagues from 1991), the present paper represents a valuable advance because it shows that one can capture the fundamental scaling of the system purely with reference to the process parameters.*

Thank you for summarizing our manuscript in a way that nicely captures our intended message.

> *I really appreciate that the authors took the time to craft an impeccable manuscript. The writing is lucid, the scope is thorough, and the arguments are well articulated. In fact, I've rarely seen such a flawless presentation in a first-time submission. Well done!*

Thank you for acknowledging the effort that we invested!

> *The decision to focus on the special case of m=1/2, n=1, while detailing the more general case in an appendix, strikes me as a sensible approach. The special case turns out to be beautifully simple. The general case is more complicated, and if they had tried to cover it in the main text the result would have been harder to penetrate. Yet they show that the*

> *fundamental insights derived from the simpler case usually hold for the more general one as well.*

Thank you for supporting this approach. Presenting the general case in an appendix was an efficient compromise between the two alternative options of presenting it in the main text of this manuscript or in a separate manuscript. The former option would have made the manuscript very hard to read, while the latter could have led readers to reasonably doubt the general validity of our results. An important, additional objective of including the general case in this manuscript was to help readers avoid conclusions that are valid only for the special case of $m = 0.5$ and $n = 1$. For instance, the characteristic height, defined as $h_c = U/K$ (Eq. 4) for the special case, should not be interpreted as expressing the competition between only uplift and incision, because in the general case it also depends on the diffusion coefficient D (Eq. A9). See further below a similar discussion regarding the characteristic time t_c.

> *Specific comments keyed to text by page / line number:*
>
> *1/8 Just an observation that we have a bit of a terminology problem as a community, which is illustrated here with the use of "Landscape Evolution Model" to refer to a set of governing equations. The word "model" is variously used to describe qualitative concepts, equation sets, numerical approximation methods and algorithms that represent certain processes, and particular computer programs that implement those numerics. Of course, I don't expect the authors to solve this problem! But given that "LEM" has been used in the past to describe numerical models, you might consider specifying "widely used equation set for landscape evolution models (LEMs)..."*

We agree, and we changed our wording in page 1, line 8, and page 2, line 14.

> *2/ 28 suggest "describes" rather than "simulates" (e.g., it would seem odd to say that mg "simulates" the force of gravity on an object at earth's surface, though if you calculated in a computer program that graphically depicts a falling rock, that might reasonably be called a simulation)*

We followed this suggestion in the revised manuscript (page 2, line 32).

> *3/8 "in keeping with the stream power law": there are some devils in the details here; maybe simplest to summarize as "in keeping with the stream power law and the common assumption that discharge per unit width scales with the square root of drainage area"*

We followed your recommendation in the revised manuscript (page 3, lines 11–12).

> *3/17-18 this is just a minor terminology issue, but for what it's worth I think of soil creep as a general transport phenomenon that can be caused by the various processes listed, and that is often described quantitative using a diffusion model. To say "diffusive process" mixes the process and the model thereof. To list soil creep as a process comparable to bioturbation somehow mixes a possible causative agent (bioturbation) with the phenomenon that results (soil creep). Consider re-wording.*

We have re-worded this sentence in a way that we hope is clearer (page 3, line 23).

> *3/23-25 nice job being clear about baselevel lowering being mathematically equivalent to "uplift", which therefore needn't be caused by some kind of odd hillslope-scale vertical tectonics!*

Thank you.

> *4/7-8 I think it would be fair to also cite Willgoose et al. 1991b (WRR, "part 2") here. I think of this work as the first serious introduction of dimensional analysis applied to landscape evolution, and he and his colleagues Ignacio Rodriguez-Iturbe and Rafael Bras deserve credit for leading the way.*

Our statement in these lines does not refer in general to dimensional analyses of LEMs. Rather, it refers specifically to dimensional analyses that explore the scales of regime transitions in LEMs that do not distinguish a priori between these regimes. The model by Willgoose et al. (1991) does distinguish between regimes and, thus, we did not cite it as an example of these specific dimensional analysis. We revised the manuscript (page 4, lines 16–17) so that it is clearer that we refer to specific cases of dimensional analyses.

However, we too view G. Willgoose and his colleagues as having led dimensional analysis of LEMs and, based on your comment, we realize that our view could be communicated more clearly. Thus, we added an additional citation of Willgoose et al. (1991) in page 2, line 20 of the revised manuscript, where we clarify what is and what is not new in our study.

> *eq 16 and following paragraph: beautiful!*

Thank you.

> *6/21 this is an odd result, which seems to neglect the fact that dimensionless drainage area can in principle range over orders of magnitude—thus, it's possible to have elements of the solution domain in which either the 1st or 2nd term on RHS dominates... which the next paragraph acknowledges.*

Based on your comment, we realized that our statement "none of the terms of this LEM (Eq. 1) can be neglected" should have included the qualifier "everywhere". Thus, we changed our statement to "none of the terms of this LEM (Eq. 1) is negligible everywhere across a landscape" and we added two sentences that further clarify this statement (page 7, lines 21–24).

> *8/12 why 34 in particular? Was this a systematic experimental design?*

No, 34 just happens to be the number of simulations that we ran. We mention this number to clarify that we did not run only the three simulations that we present in Figs. 1–5. In the revised manuscript, in page 9, line 12, we wrote "multiple combinations" instead of "34 combinations" so readers will not wonder whether this number is important. Further below, in page 9, line 15, we mention that the combinations were 34, but in a way that we hope will not give the impression that this specific number is important.

> *8/12-23 it would be helpful to state whether the model was actually run in dimensional form and then the output rescaled (which is probably the case). The sentence "We rescaled..." implies that dimensional grid coordinates were converted into dimensionless equivalents, but it is not explicitly stated whether this rescaling was applied to the inputs (i.e., input grid and K, D, U) or the outputs.*

See our response to your next comment.

> *8/25-26 same question arises here: "run on rescaled versions of the same random TIN" is ambiguous because it seems to suggest that you took a dimensional TIN and rendered it*

dimensionless before conducting each run. I'm guessing that what was actually done was to calculate ahead of time, for each set of parameters, what horizontal and spatial dimensions would produce a size of 200x400 lc and 0-0.1 hc, and then set up dimensional model runs using these spatial dimensions. Is that correct? The alternative I suppose is that the inputs were re-scaled, meaning you used units of lc, hc, and tc instead of meters and years; but if that were the case, the initial grid would be always exactly the same in horizontal point locations, with the only differences being in the vertical scale.

Yes, it is correct that we calculated ahead of time what horizontal and vertical lengths would lead to the desired domain sizes. We realized that we did not sufficiently describe how we set up the rescaled TINs, and we thank you for this question. We added a description of the procedure that we followed in Appendix C1.2 of the revised manuscript (page 30, lines 13–29) and a pointer to that description in the main text (page 9, lines 17–18). Below, we describe the workflow that we developed in some more detail than in the revised manuscript, because we assume that you would be interested to read how we used your CHILD model to implement our modeling approach.

1. Using Matlab, we generated random TINs with x-coordinates between 0 and 200, y-coordinates between 0 and 400, average mesh edge of 0.8, and z-coordinates drawn from a uniform distribution between 0 and 0.1. Thus far, these values are just numbers without any units. In other words, these TINs are dimensionless. We generated these TINs using Matlab to have control of the next steps of our procedure.

2. For each combination of the parameters K, D, and U, we calculated the characteristic length and height scales l_c and h_c (Eqs. 3 and 4). In parallel, for each parameter combination, we calculated the CHILD input parameter KB according to Eq. (C2) (denoted as k_b in that equation) and used D and U as the CHILD input parameters KD and UPRATE.

3. Using Matlab, we multiplied the x- and y-, and z-coordinates of the dimensionless TINs by the calculated characteristic length and height scales l_c and h_c, respectively. This yielded the dimensional TINs corresponding to each parameter combination. CHILD assumes that x, y, and z are in units of meters, therefore, we calculated l_c and h_c in units of meter as well. We exported the dimensional TINs as .txt files to be used as inputs by CHILD.

4. For each parameter combination, we created a .in CHILD input file that included the calculated KB, KD, and UPRATE parameters, and a pointer to the .txt file with the corresponding dimensional TIN (entered as parameter POINTFILENAME).

Note that we automated the above workflow to ensure that each parameter combination would reliably use the correct dimensional TINs. Thus, for each combination of the parameters K, D, and U, the values of the CHILD parameters KB, KD, UPRATE, and POINTFILENAME were automatically written into a .in CHILD input file that was automatically named according to that combination. Likewise, for each combination of K, D, and U, the .txt TIN file that was exported by Matlab was automatically named. Finally, the .in and .txt files corresponding to each parameter combination were automatically placed together in an automatically-named subfolder, where a call to the compiled *child* program and the corresponding .in input file was automatically issued.

> *9/1 I like how you varied lc and hc independently in these illustrations*

Thank you.

Thank you.

You are correct that the characteristic time t_c "is the time required for one unit of incision given one unit area-slope quantity." However, we would like to add some precautionary comments.

First, t_c depends only on the incision coefficient K (Eq. 5) only in the case of $m = 0.5$ and $n = 1$; in general, t_c depends on all three parameters K, D, and U (Eq. A10). Therefore, we think it would be misleading to pin an explanation of t_c exclusively on the incision term. Indeed, t_c is also the time required for one unit of deposition (or erosion) given one (or minus one) unit of curvature, and for one unit of elevation increase given one unit of uplift rate.

Furthermore, explaining t_c in terms of unit elevation change could lead to circular logic. Specifically, taking uplift as an example, the characteristic height h_c is the elevation uplifted during one unit of characteristic time t_c, which is the time needed to uplift one unit of h_c, which is… Therefore, we can only explain h_c in terms of t_c, or t_c in terms of h_c, but not both. We chose the former for reasons that we explain in the following paragraphs.

To summarize Sect. 4.1, we multiplied by t_c the rates of elevation change due to incision, diffusion, and uplift, i.e., the three RHS terms of Eq. (1), to express them in units of height. We used the resulting height scales to derive topographic relations that express competitions between processes. If we had used some other, arbitrary time interval Δt instead of t_c, we would have obtained the exact same topographic relations, because the multiplications that convert rates of elevation change into heights are all linear in time. For instance, the steady-state incision–uplift balance, i.e., the condition $K\sqrt{A}|\nabla z|\,\Delta t = U\,\Delta t$, is satisfied at points with $\sqrt{A}|\nabla z| = U/K$ and $\nabla^2 z = 0$ for any Δt.

In contrast, in the case of the horizontal direction, which we presented in Sect. 4.2, the incision time t_I scales linearly with the length scale l (Eq. 34), but the diffusion time t_D scales with l^2 (Eq. 35). Therefore, the length scale l is not eliminated from the definition of the Péclet number (Eq. 36). As it turns out, an implication of this is that at points with $l \approx l_c$ and $A \approx A_c$, where Pe ≈ 1, the incision and diffusion time scales are such that $t_I \approx t_D \approx t_c$ (page 19, line 21–23 of the revised manuscript). In other words, the horizontal advection–diffusion balance leads to conditions that give explanations to both l_c and t_c.

Consequently, focusing on the vertical direction (Sect. 4.1), we interpreted the characteristic height h_c in terms of the characteristic time t_c, which we interpreted – along with the characteristic length l_c – by focusing on the horizontal direction (Sect. 4.2). In the opening of Sect. 4.2 of the revised manuscript, we added a sentence stating that we can use the Péclet number to interpret l_c and t_c (page 18, lines 17–19).

eq 27: suggest pointing out to readers that this is just a rearrangement of the expected slope-area relationship for channels at steady state—thus, one recaptures this important relation.

We compare Eq. (27) to the slope–area relationship for steady-state channels in Sect. 4.1.3 (page 17, lines 1–8). We added a pointer to this comparison at the end of the paragraph that introduces Eq. (27) (page 15, lines 2–3).

fig 6: this is a nice illustration. If I were to change anything, it would simply be to plot -h_I (i.e., include the minus sign implied by eq 29 in relation to h_c and h_D).

Thank you for the suggestion. However, we are plotting $+h_I$ to illustrate that the h_I and h_D curves have a constant distance that is equal to h_c, i.e., to graphically illustrate Eq. (29).

15/10-17 This is a nice testable prediction. I can imagine issues arising related to the identification of drainage area (e.g., depending on the routing scheme used, at high res, many points within a valley could register as low A even though they are morphologically influenced by channelized flow), but worth trying.

Thank you. We agree that this would be an interesting research question, but for future work.

22/16-17 suggest making these separate sentences

Thank you for the suggestion. We separated these sentences in the revised manuscript (page 23, lines 30–31).

C1.2 why was it necessary to generate TINs using Matlab rather than just relying on CHILD's own TIN generation routines?

We mainly used Matlab to control and automate the preparation of input files for CHILD as described above.

2. Response to the anonymous referee

This paper contains a clever and thorough elaboration on dimensional analysis of modeled hillslope-channel interactions. I enjoyed reading it, and I expect it will be influential. The key advance is the recasting of a governing equation for landscape evolution due to uniform rock uplift, linearly slope-dependent soil creep, and stream power-dependent channel incision in a dimensionless form with no parameters, which allows the rescaling of a single solution (for a given set of initial and boundary conditions) to any set of dimensional parameters. I have a few comments for the authors to consider as they revise their manuscript, but I recommend publication without re-review.

Thank you.

The authors might consider commenting on the limits of rescaling the linear diffusion term to very steep slopes.

The linear diffusion term of Eqs. (1) and (16) cannot describe real landscapes with very steep hillslopes, even though mathematically it can take any non-negative real number as the slope. We

added a comment about this in Sect. 2.1 after the discussion of the types of landscapes which Eq. (1) is presumed to describe (page 3, line 39, to page 4, line 2).

> *What about channel width? Presumably channels in this model are assumed to be "sub-grid-scale"; how is that taken into account in the governing equation or the numerical scheme?*

We thank you for this question, because it connects our study with an important discussion that has been ongoing in the literature concerning this family of LEMs.

Specifically, Pelletier (2010) discusses the resolution-dependence of flow-routing algorithms, and thus of LEMs that use these algorithms. To minimize this dependence, Pelletier (2010) recommends including the factor δ/w in the diffusion term, where δ is the grid resolution and w is the flow width. Similarly, Perron et al. (2008) included the inverse factor, i.e., w/δ, in the incision term.

Our CHILD simulations did not include such factors. However, our rescaling approach guaranteed that the resolution-dependence was consistent and reproducible across rescaled simulations. Specifically, across rescaled landscapes, both δ and w scaled with l_c. Therefore, corresponding points of rescaled landscapes had equal δ/w ratios, and maintained consistent dependence on resolution. In the case of landscapes that are not rescaled, however, the dependence on resolution would not be consistent.

(Note that in the case of CHILD, which uses TINs, δ would correspond to some characteristic TIN widths, e.g., to lengths of edges of Voronoi polygons. Furthermore, note that, although Eq. (1) does not explicitly include flow widths, we can implicitly assume how flow widths would scale with drainage area, because the derivation of Eq. (1) is based on assumed relationships between flow width and drainage area; e.g., Tucker et al., 2001; Perron et al., 2008.)

In the revised manuscript (page 7, lines 4–18), we addressed your question by adding a discussion about various factors that control the solution of Eq. (1) in ways that are not apparent just by looking at Eq. (16), the dimensionless form of Eq. (1). Specifically, we pointed out that alternative dimensionless forms of Eq. (1) are able to reveal some properties of the LEM that are not revealed by Eq. (16). One such example is the resolution-dependence, as described above. Other examples are the dependence of solutions on boundary conditions, initial conditions, or on the size of the domain. For instance, Eq. (19) of Perron et al. (2008) includes the size of the domain (l in their notation) and the steady-state total relief (ζ); therefore, it explicitly shows that solutions depend on the size of the domain and the initial conditions (given that, for a given landscape, ζ depends on the initial conditions).

> *Section 4.1.1: I suspect that this framework could quantify the independent controls on elevation contour shapes and vertical relief noted by Howard (1997), Tucker and Bras (1998), and Perron et al. (2008). Perhaps worth discussing.*

Indeed, one could use equations of Sect. 4.1 (e.g., Eqs. (29), (30), or (31)) to show that, in the case of Eq. (1), if the characteristic height h_c is changed, while the characteristic length l_c is kept constant, then a landscape can be rescaled only vertically and not horizontally, i.e., the vertical components of the variables will be rescaled, while the horizontal components will remain constant. Note, however, that in the case of the general LEM (Eq. A1), horizontal and vertical components cannot in general be rescaled independently. The definitions of characteristic scales of the general LEM (Eqs. A8–A10) reveal the values of the exponents m and n for which horizontal, vertical, or temporal components of variables can be controlled independently, as shown, for example, in the discussion in page 25, lines 6–15.

The mechanisms controlling divide migration have been the subject of considerable discussion in the recent geomorphological literature. While exploring the properties of the incision height h_I, which is equal to the steepness index k_s in the case of Eq. (1), we happened to reproduce the result of Whipple et al. (2017), that the steepness index (among several other metrics) predicts the direction of ridge migration (it is Whipple et al. who characterize this as a result, not us). We believe this is worth reporting, without wading into the argument between those who use the steepness index and those who use χ and other metrics.

As you wrote, the direction of divide migration depends on the difference of erosion rates across the divide. Near the ridge of a landscape that follows Eq. (1), erosion is caused by both the incision and the diffusion term. Thus, we found it interesting that, in our simulations, the direction of divide migration could be predicted using the incision height h_I on its own, i.e., without taking into account the diffusion height h_D. This is in line with Willet et al.'s (2014) assumption that the reorganization of drainage basins could be studied by focusing only on river channels, leaving hillslopes out of the analysis. However, this interesting observation is beyond the scope of the present study and, thus, we did not include it in our manuscript, because we did not want to assert something that we have not proved.

Section 4.2.2: The use of the flow path length (which depends on topography) as the horizontal length scale in the spatially variable Peclet number seems to run counter to the spirit of the non-dimensionalization that is the paper's centerpiece, in which the authors avoided using any length scales that are topographic outcomes of the model evolution. I understand the authors' reason for doing the calculation this way (Fig. 11), but it does make it difficult to use this definition of the Peclet number to predict model outcomes.

(Note that the subsection that follows Sect. 4.2.2 was mistakenly numbered as 4.2.4 in the submitted manuscript. In the revised manuscript, we corrected this mistake, and in this response, we refer to this section as 4.2.3.

We thank you for this comment, because it gives us an opportunity to clarify the motivation behind our key assumptions and methodologies.

Using the flow path length to calculate values of the Péclet number does not run counter to the spirit of our analysis. We avoided using topography-dependent scales to define the characteristic scales. To interpret the characteristic scales, however, we introduced process-specific scales that vary across the landscape, such as the incision and diffusion heights h_I and h_D (Eqs. 21 and 22). Then, we explored whether there is anything special about the points at which the process-specific scales become equal to the characteristic scales or to each other.

As we explain further above in our response to referee Greg Tucker, we found that we could interpret either the characteristic height h_c in terms of the characteristic time t_c, or t_c in terms of h_c, but not both. We found, however, that we could interpret t_c together with the characteristic length l_c by including both the drainage area A and a length scale l in the definition of the incision time t_I (Eq. 34). This revealed that advective and diffusive propagation of perturbations lead to different scalings between

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time and length scales. Therefore, we could interpret t_c and l_c in terms of the competition between advective and diffusive propagation of perturbations, and, in turn, interpret h_c in terms of t_c.

Thus far, l could be any arbitrary length scale. However, to calculate values of the Péclet number across a landscape, a specific length scale should be chosen. In Sect. 4.2.2 we chose flow path length to calculate Péclet number values for Fig. 10, a figure which suggests that defining the Péclet number according to Eq. (36) is reasonable, while in Sect. 4.2.3 we discussed the similarities and differences between our Péclet number definition and that of Perron et al. (2008, 2009, 2012). The Péclet number, like the incision time t_I, cannot be (usefully) defined in terms of the model parameters alone.

Generally, we think that there is no single Péclet number definition that fits all problems. Rather, employing different definitions that use different length scales, we can study different properties of landscapes. This is analogous to studying turbulence by using different characteristic system sizes for different kinds of flows, e.g., flow in pipes vs. flow past a cylinder.

In the case of landscapes, one could study advective and diffusive propagation of perturbations using the flow path length, as discussed in our manuscript, but one could also characterize sizes of whole landscapes – relative to their parameters K and D – using the total size of each landscape as the length scale and defining a landscape-wide Péclet number. Small values of such a Péclet number would correspond to landscapes that consist mainly of hillslopes, because they are too small relative to the spacing of valleys that would be expected for the parameters of these landscapes. In contrast, large values of such a Péclet number would correspond to dissected landscapes, which are large enough to fit their valleys. The transition between these two cases need not occur at a Péclet number value of roughly one; this value would need to be discovered using numerical experiments (e.g., Perron et al., 2012).

To summarize our response to your comment, a key motivation behind Sect. 4.2 is to interpret the characteristic time t_c independently of the characteristic height h_c. For this, we defined an incision time t_I whose scaling with the length scale l is different than that of the diffusion time t_D (Eqs. 34 and 35). To illustrate that these two time scales define a reasonable Péclet number, we used the flow path length, which is a reasonable scale because it is in line with the way that perturbations are propagated by the incision and diffusion terms. Given that we modified Perron et al.'s (2008, 2009, 2012) original definition of the Péclet number, we presented a brief comparison of the two definitions, but a more thorough comparison would be beyond the scope of this study.

As we wrote further above in our response to referee Greg Tucker, we added a sentence stating that we can use the Péclet number to interpret l_c and t_c in page 18, lines 17–19.

> *The summary and conclusions section is a bit long.*

We agree that it is long, but we think that this is a feature rather than a bug: a manuscript with such length, complexity, and number of results, requires a long summary to be sufficiently summarized.

> *The inclusion of the dimensional analysis for the more general case of 2m != n is nice. Although it is understandably relegated to an appendix, this analysis considerably broadens the applicability of the paper.*

Thank you. See also our response to referee Greg Tucker regarding this appendix. As we explain there, relegating this analysis in the appendix was a compromise, which came with certain trade-offs. For instance, some conclusions of this analysis will inevitably receive less attention than they might otherwise receive.

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Scaling and similarity of a stream-power incision and linear diffusion landscape evolution model

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Abstract. Scaling and similarity of fluvial landscapes can reveal fundamental aspects of the physics driving their evolution. Here we perform dimensional analysis on the governing equation of a widely used landscape evolution model (LEM) that combines stream-power incision and linear diffusion laws. Our analysis assumes that length and height are conceptually

10   distinct dimensions, and uses characteristic scales that depend only on the model parameters (incision coefficient, diffusion coefficient, and uplift rate) rather than on the size of the domain or of landscape features. We use a previously defined characteristic length, height, and time scales, but, for the first time, we combine all three in a single analysis. introduce new characteristic height and time scales. We use Using these characteristic scales, to we non-dimensionalize the LEM, such that it includes only dimensionless variables and no parameters. This significantly simplifies the LEM by removing all

15   parameter-related degrees of freedom. The only remaining degrees of freedom are in the boundary and initial conditions. Thus, for any given set of dimensionless boundary and initial conditions, all simulations, regardless of parameters, are just re-scaled copies of each other, both in steady state and throughout their evolution. Therefore, the entire model parameter space can be explored by temporally and spatially re-scaling a single simulation. This is orders of magnitude faster than performing multiple simulations to span multi-dimensional parameter spaces.

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The characteristic length, height, and time scales are geomorphologically interpretable; they define relationships between topography and the relative strengths of landscape-forming processes. The characteristic height scale specifies how drainage areas and slopes must be related to curvatures for a landscape to be in steady state, and leads to methods for defining valleys, estimating model parameters, and testing whether real topography follows the LEM. The characteristic length scale is

25   roughly equal to the scale of the transition from diffusion-dominated to advection-dominated propagation of topographic perturbations (e.g., knickpoints). We introduce a modified definition of the landscape Péclet number, which quantifies the relative influence of advective versus diffusive propagation of perturbations. Our Péclet number definition can account for the scaling of basin length with basin area, which depends on topographic convergence versus divergence.

1   Introduction

30   Hillslopes and river valleys are organized in striking patterns that appear to be repeated across landscapes and scales. Furthermore, within each landscape the transition from hillslopes to valleys seems to occur at a characteristic scale. These two properties have captivated scientists from the early days of geomorphology (e.g., Gilbert, 1877; Davis, 1892). Both

properties are thought to be related to the scaling of processes that shape fluvial landscapes (e.g., Perron et al., 2008, 2012; Paola et al., 2009).

Scaling problems are often studied with the aid of dimensional analysis (e.g., Sonin, 2001; Bear and Cheng, 2010) which
5 stems from Fourier's principle that all terms of physically meaningful equations should have consistent dimensions (Huntley, 1967). Dimensional analyses of landscape evolution models (LEMs) have been used to describe how the relative strengths of landscape-forming processes control properties of ridge and valley topography, such as drainage density (Willgoose et al., 1991), shapes of basins (Howard, 1994), fluvial relief of mountains (Whipple and Tucker, 1999), topographic roughness (Simpson and Schlunegger, 2003), valley spacing (Perron et al., 2008, 2009), and drainage areas of first- and second-order
10 valleys (Perron et al., 2012). The aforementioned studies used LEMs that, while differing in details, all assumed that fluvial landscapes are shaped by a combination of advective and diffusive erosion, with the former dominating valleys and the latter dominating hillslopes.

Here we present a dimensional analysis of the governing equation of a simple, widely used LEM (see Eq. 1 below). Our
15 work ~~departs from previous dimensional analyses of LEMs in two ways~~ is based on two key premises. First, we define characteristic scales from the model parameters, rather than from extrinsic properties of the simulated landscape, such as the domain size or relief. Characteristic scales defined in this way are intrinsic to each landscape and its parameters (and thus to its underlying properties and to the strengths of the processes that shape it), and are independent of the initial and boundary conditions of any simulation. Second, in our approach we distinguish between the dimensions of horizontal length and
20 vertical height. These two ~~assumptions~~ premises are not new (e.g., Willgoose et al., 1991; Whipple and Tucker, 1999; Perron et al., 2008; Robl et al., 2017), but here, for the first time, we apply them jointly to ~~the definitions, and the interpretations, of~~ define and interpret all of the characteristic scales in this LEM. In so doing, we obtain three characteristic scales – a characteristic length, height, and time – that significantly simplify the model's governing equation. These characteristic scales are also geomorphologically interpretable, linking competition between processes to the scales of the features that
25 emerge from these processes.

Our specific results directly apply only to the LEM that we dimensionally analyzed and to the landscapes that we can assume to be described by this LEM. However, our work illustrates an approach for the definition and interpretation of characteristic scales that could potentially be employed in dimensional analyses of other LEMs as well.

30 **2 Theory**

2.1 Landscape evolution model

We perform dimensional analysis of a simple, widely used model that ~~simulates~~ describes the evolution of landscapes under the influence of stream-power incision, linear diffusion, and uplift, according to the governing equation

$$\frac{\partial z}{\partial t} = -K\sqrt{A}|\nabla z| + D\nabla^2 z + U \, , \tag{1}$$

(e.g., Howard, 1994; Dietrich et al., 2003) where $z(x, y)$ is the elevation of a point with horizontal coordinates (x, y), t is time
35 and $\partial z/\partial t$ is the rate of change of elevation, K is the coefficient of incision, A is the drainage area (per unit contour width) at

2

the point (x, y), and $|\nabla z|$ is the norm of the gradient of z (i.e., the topographic slope in the direction of steepest descent, to which the gradient vector points by definition), D is the coefficient of diffusion and $\nabla^2 z$ is the Laplacian of z (i.e., the topographic curvature, here assumed to be positive in concave-up areas, e.g., valleys, and negative in concave-down areas, e.g., hillslopes), and U is the uplift rate. The dimensions of the variables and parameters of Eq. (1) are discussed in the

5 following subsection, which focuses on dimensional analysis.

The incision term $-K\sqrt{A}|\nabla z|$ gives the rate of change of elevation due to detachment-limited sediment transport by flowing water, assumed to be proportional to stream power, i.e., the work performed by water per unit time, per unit streambed area (Dietrich et al., 2003). The incision term of Eq. (1) is a specific case of the more general incision term $-KA^m|\nabla z|^n$, which

10 can take on different drainage area and slope exponents m and n in order to express more general incision behavior (e.g., Dietrich et al., 2003). Here we assume m=0.5 and n=1, in keeping with the stream power law, and with the common assumptions that discharge scales linearly with drainage area and that channel width scales with the square root of drainage area, but in Appendix A we present results from dimensional analysis of the same LEM with generic exponents m and n (Eq. A1). We show that all the results and interpretations that we derived from the simplified LEM (Eq. 1), which we present

15 in the main text, can be recovered from results of the generic LEM (Eq. A1) by setting m=0.5 and n=1. We do not present any results or interpretations that do not satisfy this condition. In other words, our results are not merely properties of a special case; we just describe them by simpler formulas whose presentation and interpretation is straightforward. Additionally, the incision term could include a threshold, below which no incision occurs (e.g., Dietrich et al., 2003; Perron et al., 2008). We investigate the scaling behavior of an LEM with an incision term with a non-zero incision threshold in other

20 work, to be published separately.

The diffusion term $D\nabla^2 z$ gives the rate of change of elevation due to diffusive sediment transport processes, such as soil creep, due to bioturbation, freeze–thaw cycles, etc. The diffusion term is linear and assumes that the flux of sediment is proportional to the topographic slope (Fernandes and Dietrich, 1997). Depending on the sign of the curvature, this term can

25 be either negative or positive, corresponding to erosional diffusion on ridges and hillslopes or depositional diffusion in valleys.

The term U gives the rate of increase of elevation relative to the boundary due to either a falling base level or tectonic uplift of the domain's interior. Throughout this paper, we refer to both effects as uplift since they are mathematically

30 indistinguishable.

If a landscape evolves to a condition in which, at every point, the three terms of the right-hand side of Eq. (1) cancel each other out, then $\partial z/\partial t = 0$ everywhere; this condition is termed steady state or dynamic equilibrium. Numerical simulations are typically assumed to have converged to steady state when the rate of elevation change is everywhere smaller than an

35 arbitrary, small threshold ε:
$$| \partial z/\partial t | \le \varepsilon \quad \forall(x, y) . \tag{2}$$

Equation (1) is presumed to describe soil-mantled landscapes with sufficiently cohesive soils and gentle slopes, where we can assume that incision is detachment-limited and diffusion is linear (e.g., Perron et al., 2008). Other types of landscapes

are shaped by processes that cannot be described by Eq. (1). For example, diffusive processes on steep soil-mantled hillslopes are better described by a non-linear diffusion term (e.g., Roering et al., 1999, 2007).

Equation (1) assumes that all three processes act at all points of the landscape, without distinguishing between channels or hillslopes (e.g., Howard, 1994; Simpson and Schlunegger, 2003). All three terms are needed to model fluvial landscapes. Uplift fulfills the role of the source term, forcing the evolution of the landscape (e.g., Tucker and Hancock, 2010). Without it, the landscape would decay to a flat surface of zero elevation. The combination of the incision and diffusion terms is necessary for the emergence of ridges and valleys. Whereas the incision term amplifies topographic perturbations, setting in motion a positive feedback, the diffusion term dampens them, leading to a negative feedback (e.g., Smith and Bretherton, 1972; Perron et al., 2012). Both types of feedback are needed for the synthesis of surfaces with complex structures that resemble ridge and valley topography; therefore, the incision and diffusion terms of Eq. (1), or other terms with equivalent properties, represent the simplest combination of processes that can model landscapes characterized by ridge and valley topography (e.g., Smith and Bretherton, 1972; Howard, 1994). Because different points have different topographic properties (drainage area, slope, and curvature), the modeled incision and diffusion processes have different relative strengths across the landscape. Thus, even though distinct convergent (channel) and divergent (hillslope) landforms are not specified a priori, they can emerge from Eq. (1) (e.g., Howard, 1994), at scales that can be explored using Ddimensional analysis has been used to study how and at which scales these regimes emerge (e.g., Perron et al., 2008, 2009, 2012).

2.2 Dimensional analysis

Our dimensional analysis of Eq. (1) begins by specifying the dimensions of its various variables. We will rescale Eq. (1) in the horizontal direction separately from the vertical direction so that we can study scaling of lengths and reliefs separately (e.g., Dietrich and Montgomery, 1998). Therefore, we must assume that the coordinates of points (x, y) have dimensions of length (L), while elevation z has an equivalent, but conceptually distinct, dimension of height (H). Huntley (1967) outlined a theoretical justification for distinct dimensions for quantities with identical units, and presented examples demonstrating the benefits of this approach. Distinct dimensions of length and height have been adopted by some previous studies of scaling of landscapes (e.g., Willgoose et al., 1991), but using different approaches than the one outlined below. The three fundamental dimensions in this model are length (L), height (H) and time (T), and all variables of Eq. (1) have dimensions that are powers, products, or ratios of L, H, and T. Specifically, the rate of elevation change $\partial z / \partial t$ and, thus, also the incision, diffusion, and uplift terms of Eq. (1) have dimensions of H T^{-1}. Given that the gradient $|\nabla z|$ and the curvature $\nabla^2 z$ have dimensions of H L^{-1} and H L^{-2}, respectively, the parameters K, D, and U must have dimensions of T^{-1}, L^2 T^{-1}, and H T^{-1}, respectively.

Because the dimensions of all variables of Eq. (1) can be expressed in terms of L, H, and T, we can non-dimensionalize Eq. (1) using a characteristic length l_c, a characteristic height h_c, a characteristic time t_c, and combinations thereof. For important reasons that are discussed throughout this study, we choose to define l_c, h_c, and t_c as intrinsic scales, i.e., in terms of the model's parameters K, D, and U, not in terms of sizes of geomorphic features, such as basin length (e.g., Willgoose et al., 1991; Whipple and Tucker, 1999) or total relief (e.g., Whipple and Tucker, 1999; Perron et al., 2008), or in terms of the extent of the solution domain (e.g., Simpson and Schlunegger, 2003).

We define the characteristic length scale as

$$l_c := \sqrt{D/K}, \tag{3}$$

which has dimensions of length. Perron et al. (2008) showed that $\sqrt{D/K}$ is related to the competition between incision and diffusion, and that it controls the scales of valley spacing (Perron et al., 2009) and valley-network branching (Perron et al., 2012). The quantity $\sqrt{D/K}$ has also been previously used to non-dimensionalize LEMs (e.g., Duvall and Tucker, 2015).

5

We introduce a characteristic height scale

$$h_c := U/K, \tag{4}$$

which has dimensions of height, and a characteristic time scale

$$t_c := 1/K, \tag{5}$$

which has dimensions of time (these characteristic scales are equivalent to those presented by Robl et al. (2017) using a somewhat different formulation). The U/K ratio has been previously used in several different contexts. For instance,

10 $(U/K)^{1/n}$, where n is the slope exponent, is the steady-state value (e.g., Moglen and Bras, 1995; Sklar and Dietrich, 1998) of the steepness index (e.g., Whipple, 2001), which has been used to analyze river profiles and predict ridge migration dynamics (e.g., Wobus et al., 2006; Harkins et al., 2007; Whipple et al., 2017). It has also been previously noted that the U/K ratio scales relief (e.g., Sklar and Dietrich, 1998; Perron and Royden, 2013; Willett et al., 2014). To the best of our knowledge, however, previous dimensional analyses have not used U/K has not previously been used as a characteristic

15 height scale to non-dimensionalize landscape evolution equations.

The characteristic length, height, and time scales l_c, h_c, and t_c, as defined above (Eqs. 3–5), have not been previously combined to non-dimensionalize landscape evolution equations. We proceed with l_c, h_c, and t_c as defined above Here, we adopt them as a group on purely dimensional grounds, because Eqs. (3), (4), and –(5) define the only combinations of D, K,

20 and U that yield dimensions of L, H, and T, respectively. However, in Sect. 4 we also show that these characteristic scales also have interesting geomorphological properties that further justify defining them according to Eqs. (3)–(5). (

In Appendix A, we define analogous characteristic scales for an LEM with generic exponents m and n.) In this more general case, each of the three characteristic scales depends on all three parameters K, D, and U.

25

Using the characteristic scales l_c, h_c, and t_c, we can express length, height, and time in equivalent dimensionless forms:

$$x = l_c x^*, \qquad y = l_c y^*, \qquad z = h_c z^*, \qquad t = t_c t^*, \tag{6}$$

where the starred variables are dimensionless. Likewise, we can express each term of Eq. (1) as a corresponding dimensionless term multiplied by characteristic scales (or their products and ratios) which carry the corresponding dimensions.

30

Specifically, each differential (here using dz as an example) can be re-expressed as

$$dz = h_c dz^*. \tag{7}$$

Consequently, we can express the rate of elevation change as

$$\frac{\partial z}{\partial t} = \frac{h_c}{t_c}\frac{\partial z^*}{\partial t^*} = U\frac{\partial z^*}{\partial t^*}\,, \tag{8}$$

which suggests that we can view the uplift rate U as a characteristic rate of elevation change. Furthermore, we can express the gradient operator as

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} = \frac{1}{l_c}\left(\frac{\partial}{\partial x^*}\mathbf{i} + \frac{\partial}{\partial y^*}\mathbf{j}\right) = \frac{1}{l_c}\nabla^*\,, \tag{9}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the direction of the x and y coordinates and ∇^* is the gradient operator in dimensionless coordinates. Therefore, we can express topographic slope as

$$|\nabla z| = \frac{1}{l_c}|\nabla^*(h_c z^*)| = G_c|\nabla^* z^*|\,, \tag{10}$$

5   where G_c is a characteristic gradient defined as

$$G_c := \frac{h_c}{l_c} = \frac{U}{\sqrt{DK}}\,. \tag{11}$$

Our characteristic gradient G_c should not be confused with the critical slope S_c used in a non-linear diffusion law (e.g., Roering et al., 1999). Likewise, we can express topographic curvature as

$$\nabla^2 z = \frac{1}{l_c^2}\,\nabla^{*2}(h_c z^*) = \kappa_c\nabla^{*2} z^*\,, \tag{12}$$

where κ_c is a characteristic curvature defined as

$$\kappa_c := \frac{h_c}{l_c^2} = \frac{G_c}{l_c} = U/_D\,. \tag{13}$$

Our characteristic curvature should not be confused with the contour curvature, also denoted as κ_c elsewhere (e.g., Perron et
10   al., 2012). The negative of U/D has been previously shown to describe the steady-state curvature of hilltops and drainage divides (e.g., Roering et al., 2007; Perron et al., 2009). Finally, given that areas scale as the square of lengths, we can express drainage area as (e.g., Perron et al. 2008, 2012)

$$A = l_c^2 A^* = A_c A^*\,, \tag{14}$$

where A_c is a characteristic area, defined as

$$A_c := l_c^2 = D/_K\,. \tag{15}$$

15   Substituting Eqs. (6), (8), (10), (12), and (14) into the governing equation (Eq. 1) yields

$$(h_c\partial z^*)/(t_c\partial t^*) = -K\sqrt{A_c A^*}\,G_c|\nabla^* z^*| + D\,\kappa_c\nabla^{*2} z^* + U\,,$$

which can be simplified to the dimensionless form:

$$\frac{\partial z^*}{\partial t^*} = -\sqrt{A^*}|\nabla^* z^*| + \nabla^{*2} z^* + 1\,. \tag{16}$$

Equation (16) includes only dimensionless variables and no parameters. Because Eq. (16) has no parameters to be adjusted, for a given set of boundary and initial conditions, it will have only one steady-state solution, which will be arrived at via only
20   one path of evolution. This implies that all simulated landscapes with any parameters (but properly rescaled domains, boundary conditions, and initial conditions) will evolve as rescaled replicas of each other, because they can all be reduced to Eq. (16) through rescaling by the characteristic scales l_c, h_c, and t_c. We explore this rescaling property in length in Sect. 3 and in Appendix B.

Alternative dimensionless forms of Eq. (1) can reveal properties of the LEM that are not revealed by Eq. (16). For example, Perron et al.'s (2008) Eq. (19) was derived using the domain half-width as a characteristic length and the steady-state maximum relief as a characteristic height. In this way, Perron et al.'s (2008) dimensionless equation includes information about the domain size and the initial conditions (which influence the final relief; e.g., Howard, 1994); therefore, that equation highlights the dependence of its solutions on the domain size and the initial conditions. Perron et al.'s (2008) Eq. (19) is equivalent to our Eq. (16) if one can express the domain size and relief in terms of l_c and h_c, but the relationships between these two different pairs of scales will vary for different landscape configurations arising from different initial and boundary conditions.

Likewise, Eq. (16) does not reveal that flow-routing algorithms, and thus LEM solutions, can be resolution-dependent if the channel width w is smaller than the mesh resolution δ (e.g., Pelletier, 2010). This dependence can be minimized by including the factor δ/w in the diffusion term (e.g., Pelletier, 2010) or the factor w/δ in the incision term (e.g., Perron et al., 2008). Equation (16) does not include such factors. However, the rescaling of domains (detailed in Sect. 3) guarantees that both w and δ scale with l_c; this guarantees that the resolution-dependence of model solutions is consistent across rescaled landscapes.

The fact that Eq. (16) includes no parameters has an additional important implication. One can use the factors that appear in front of terms of a dimensionless equation to infer the relative importance of each term (e.g., Huntley, 1967). In the case of Eq. (16), all such factors are equal to one, which implies that none of the terms of this LEM (Eq. 1) ~~can be neglected~~is negligible everywhere across a landscape. In other words, each term may be dominant at some points of a landscape, depending on the local values of drainage area A, slope $|\nabla z|$, and curvature $\nabla^2 z$, even if it is negligible elsewhere. Therefore, none of the terms of Eq. (1) can be dropped purely on grounds of process dominance.

We can rewrite Eq. (16) in a form that reveals what controls the relative dominance of each process across a landscape. Specifically, the dimensionless quantities of Eq. (16) are equal to the ratio of the corresponding dimensional quantities over appropriate characteristic scales (Eqs. 8, 10, 12, and 14). Therefore, we can rewrite Eq. (16) as

$$\frac{\partial z/\partial t}{U} = -\frac{\sqrt{A}}{\sqrt{A_c}}\frac{|\nabla z|}{G_c} + \frac{\nabla^2 z}{\kappa_c} + 1 \,. \tag{17}$$

Equation (17) is exactly equivalent to Eq. (16) but helps illuminate different properties of the model. Specifically, Eq. (17) shows that the relative contributions of each of the topographic properties A, $|\nabla z|$, and $\nabla^2 z$ are equal to their ratios over the corresponding characteristic scales A_c, G_c, and κ_c. The ratios in Eq. (17), or other, equivalent groupings of variables and parameters, could be defined as dimensionless numbers. Often, dimensional analyses use dimensionless numbers to express the relative contributions of processes. In the case of landscape evolution models, examples of such dimensionless numbers are uplift numbers (e.g., Whipple and Tucker, 1999) and Péclet numbers (e.g., Perron et al., 2008). Equation (17) expresses how such dimensionless numbers emerge from the dimensionless governing equation.

3   Scaling and similarity of landscapes

The dimensionless form of the governing equation (Eq. 16) implies that landscapes with any parameters, but properly rescaled boundary and initial conditions (see immediately below what we term as "proper" rescaling), will evolve in such a way that snapshots of these landscapes at properly rescaled moments in time will be (horizontally and vertically) rescaled copies of each other. In other words, the evolution of these landscapes will obey temporal and geometric similarity. This, in turn, implies that such landscapes will reach geometrically similar steady states.

We consider domains, elevations, and time to be properly rescaled if they are equivalent when normalized by the characteristic length, height, and time scales l_c, h_c, and t_c, respectively. For instance, let a landscape have parameters K, D, and U, and a second landscape have parameters K', D', and U'. Variables and characteristic scales of the second landscape are primed to match the notation of its parameters. Domains are properly rescaled when pairs of points with dimensional coordinates (x, y) and (x', y') correspond to the same point with dimensionless coordinates (x^*, y^*). Thus,

$$x'^* = x^* \iff x'/l_c' = x/l_c \iff x' = (l_c'/l_c)\, x \,, \tag{18 a}$$

and, likewise,

$$y' = (l_c'/l_c)\, y \,. \tag{18 b}$$

Likewise, elevations are properly rescaled when dimensional elevations z and z' at equivalent points (x, y) and (x', y') correspond to the same dimensionless elevation z^* at (x^*, y^*), such that

$$z' = (h_c'/h_c)\, z \,, \tag{18 c}$$

and two moments in time t and t' are properly rescaled when they correspond to the same moment in dimensionless time t^* such that

$$t' = (t_c'/t_c)\, t \,. \tag{18 d}$$

We should point out that simulations of these two landscapes will reach geometrically similar steady states only if we rescale the threshold ε in the steady-state criterion of Eq. (2). Specifically, we assume that the two landscapes have reached numerical steady states that satisfy the criteria $|\, \partial z/\partial t \,| \le \varepsilon$ and $|\, \partial z'/\partial t' \,| \le \varepsilon'$. Using Eq. (8) we see that in the dimensionless coordinate system of Eq. (16) these criteria become $|\, U(\partial z^*/\partial t^*) \,| \le \varepsilon \iff |\, (\partial z^*/\partial t^*) \,| \le \varepsilon/U := \varepsilon^*$ and, likewise, $|\, U'(\partial z^*/\partial t^*) \,| \le \varepsilon' \iff |\, (\partial z^*/\partial t^*) \,| \le \varepsilon'/U' := \varepsilon'^*$. If the two numerical steady states are geometrically similar, then they must be satisfying the same dimensionless criterion, i.e., $\varepsilon'^* = \varepsilon^*$, which leads to a steady-state threshold rescaling formula:

$$\varepsilon'/U' = \varepsilon/U \iff \varepsilon' = (U'/U)\, \varepsilon \,. \tag{19}$$

In the following subsections we use a numerical model to demonstrate the temporal and geometric similarity of rescaled landscapes that is implied by the dimensionless governing equation (Eq. 16). In addition, in Appendix B we outline a simple analytical proof of this similarity property. That proof suggests that rescaling works only if we rescale initial conditions (elevations) by h_c. This implies that rescaling works only if l_c and h_c are defined ~~independently of each other~~separately, i.e., only if we assume distinct dimensions for lengths and heights.

8

3.1 Numerical demonstration

3.1.1 Model set-up

We used the Channel-Hillslope Integrated Landscape Development (CHILD) model (Tucker et al., 2001) to numerically demonstrate the similarity property revealed by the dimensionless governing equation (Eq. 16). We chose CHILD due to its wide use by the geomorphologic community, and due to the fact that it uses triangular irregular networks (TIN), which avoid the geometric bias of regular grids (e.g., Braun and Sambridge, 1997). We selected CHILD modules and parameters such that it would simulate Eq. (1). Specifically, we selected CHILD's detachment-limited incision module, with constant, uniform precipitation, along with linear diffusion and uniform uplift (see Tucker et al., 2001 and Tucker, 2010 for definitions of CHILD's assumptions, modules, and parameters). In Appendix C we present in more detail how we set up our CHILD simulations and how we retrieved our results from CHILD's output files.

We ran simulations using ~~34~~ multiple combinations of the model parameters K, D, and U. We chose baseline parameter values of $K = 10^{-6}$ a^{-1}, $D = 10^{-2}$ m^2 a^{-1}, and $U = 10^{-4}$ m a^{-1}, ~~that~~ which are typical in the literature (e.g., Perron et al., 2008; Tucker, 2009; Clubb et al., 2016), and we varied each parameter by two orders of magnitude around its baseline value, in a total of 34 parameter combinations.

We applied these parameter combinations on rescaled copies of ~~a~~ two random, dimensional TIN~~s~~. In Appendix C1.2 we describe how we prepared the rescaled TINs. Domain size was 200 l_c by 400 l_c with an average mesh edge of 0.8 l_c, resulting in approximately 150,000 ~~points~~ TIN vertices. Initial elevations were a uniform white noise, ranging between 0 and 0.1 h_c. For each simulation we calculated l_c and h_c according to Eqs. (3) and (4), respectively. We rescaled the horizontal TIN coordinates and initial elevations according to Eqs. (18 a–c).

Simulation time step lengths were not explicitly rescaled. Rather, we defined simulation time step lengths using Courant-Friedrichs-Lewy criteria (Refice et al, 2012) as described in Appendix C (Eq. C3). As seen in Eq. (C4), it turns out that the resulting time step lengths were in effect rescaled by t_c due to the dependence of the Courant-Friedrichs-Lewy criteria on ~~the~~ rescaled variables.

We ran ~~the~~ simulations until ~~numerical steady states were~~ they reached numerical steady states, which we defined using rescaled steady-state thresholds according to Eq. (19). We compared the resulting landscapes during their evolution and at their steady states.

3.1.2 Numerical results

The numerical results confirmed the rescaling properties of the dimensionless governing equation (Eq. 16). Simulations which were run on rescaled versions of the same random TIN evolved similarly in space and time. Specifically, ~~A~~at time steps rescaled by t_c (Eq. 18 d), elevations of corresponding points across simulations could be rescaled by h_c (Eq. 18 c) Furthermore, if steady-state criteria were rescaled according to Eq. (19), then simulations reached geometrically similar steady states.

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In Figs. 1 and 2 we present steady-state results for three landscapes (our baseline and two alternatives), and in Figs. 3–5 we present transient results during their evolution. These figures illustrate the geometric and temporal similarity of these landscapes. The parameter combinations of these three landscapes are a subset of all the combinations that we used; their values can be seen in Table 2. We are presenting these specific combinations for demonstration purposes as they lead to wide ranges of l_c and h_c. However, all the other parameter combinations that we tested also yielded landscapes that exhibited temporal and geometric similarity.

In Fig. 1 we show steady-state shaded relief maps, and in Fig. 2 we show elevation maps and transects. In both figures l_c and h_c increase from left to right. We vary l_c and h_c separately; thus, the characteristic gradient G_c does not vary monotonically from left to right. In Fig. 2, the thick black lines in the top panels mark transects corresponding to the elevation profiles in the bottom panels, and pass through the highest peaks of the simulated landscapes. The coloring and labeling of Figs. 1 and 2 highlight both the large differences of scale and the geometric similarity of the three rescaled landscapes. For comparison, lengths and elevations on axes and colorbars are shown both in units of km or m using bold fonts, and in units of l_c or h_c using normal fonts. Note that the l_c and h_c values for different landscapes are different as they depend on the model parameters. Color scales of elevation maps in Fig. 2 are rescaled by h_c to assist with comparing the elevations of features.

In the shaded relief maps of Fig. 1, the spatial pattern of ridges and valleys is identical across the three landscapes, illustrating their horizontal geometric similarity, although their shaded relief contrast varies, reflecting their different characteristic gradients G_c. Likewise, in the elevation maps of Fig. 2, the spatial pattern of colors is identical across the three landscapes. This illustrates that the three landscapes are geometrically similar both horizontally and vertically, because the color scales are rescaled by h_c. Finally, the horizontal and vertical geometric similarity of the three landscapes is illustrated also by the shapes of the transects of Fig. 2.

The geometric similarity of the three steady-state landscapes is exact, not just visually convincing. Our domain rescaling procedure does not affect the point IDs of the TIN vertices, so we can directly compare corresponding points using their IDs. In these simulated landscapes, the maximum absolute difference of dimensionless elevations z^* of corresponding points was less than 10^{-9} units of h_c.

Figures 3–5 show shaded relief maps, elevation maps, and transects for four snapshots in time during the evolution of the three landscapes. Each column shows snapshots of the three landscapes that correspond to the same moment in dimensionless time (but different moments in dimensional time), with time increasing from left to right. Each row shows the evolution of one landscape, with one set of model parameters. As in Figures 1 and 2, lengths and elevations on axes and colorbars in units of km or m are shown in bold, and the corresponding values in units of l_c or h_c are shown in normal font for comparison. Labels above each snapshot show time in millions of years (in bold fonts) and in units of t_c (in normal fonts). Color scales of elevation maps in Fig. 4 are rescaled by h_c to assist with comparing the elevations of features. For each landscape, we use one color scale that remains constant in time to highlight how relief evolves. Each landscape's color scale is set to match the highest elevation among the four snapshots. We use different color scales for different landscapes and rescale them by h_c to facilitate comparison of features across landscapes. Visual comparison shows the three landscapes to be temporally similar, since at the same moments in properly rescaled time they are in geometrically similar transient states. Note that l_c and h_c are different across landscapes, but for each landscape, they are constant in time. Note also that the

10

snapshots that we present are not equally spaced in time, because these landscapes evolve rapidly at first, and much more slowly later.

As in the case of steady-state landscapes, the temporal and geometric similarity of the three evolving landscapes is exact, not just visually convincing. Throughout the evolution of the three landscapes, the maximum absolute difference of dimensionless elevations z^* of corresponding points at corresponding moments of dimensionless time is less than 10^{-7} units of h_c.

3.2 Implications of temporal and geometric similarity

3.2.1 Deducing how model parameters control landscape metrics

The geometric similarity of rescaled landscapes implies that all horizontal coordinates (x, y) and elevations z, and thus all lengths and heights, will be rescaled by the characteristic length and height scales l_c and h_c, respectively, according to Eqs. (18 a–c). Likewise, the temporal similarity of the evolution of rescaled landscapes implies that all time intervals will be rescaled by the characteristic time scale t_c according to Eq. (18 d). Thus, any variables that combine the dimensions of length L, height H, and/or time T will be rescaled by the corresponding combinations of l_c, h_c, and t_c. For example, as we showed in Sect. 2.2, drainage areas, slopes, curvatures, and rates of elevation change are rescaled by the characteristic area $A_c = l_c^2$, characteristic gradient $G_c = h_c/l_c$, characteristic curvature $\kappa_c = h_c / l_c^2$, and uplift rate $U = h_c/t_c$ (we showed that U can be viewed as a characteristic rate of elevation change; Eq. 8).

The characteristic length, height, and time scales l_c, h_c, and t_c depend only on the model parameters K, D, and U (Eqs. 3–5). Therefore, we can infer how any variable scales with K, D, and U based on how its corresponding characteristic scale combines l_c, h_c, and t_c (which we can infer from how the variable's dimensions combine L, H, and T). For example, from the definitions of l_c, G_c, h_c, t_c, A_c, κ_c, and U (Eqs. 3, 11, 4, 5, 15, 13, 8) we infer that if we change the incision coefficient by a factor k, i.e., if we change it from K to kK, then all distances and slopes will change by $1/\sqrt{k}$, all reliefs, durations, and drainage areas will change by $1/k$, and all (Laplacian) curvatures and rates of elevation change will remain the same.

As an additional example, the ratio of the characteristic length l_c to the characteristic time t_c defines a characteristic horizontal velocity

$$u_c := \frac{l_c}{t_c} = \sqrt{DK} \,. \tag{20}$$

We can, thus, deduce that horizontal velocities of drainage divide migration must scale by u_c, i.e., by $\sqrt{DK}$. This does not imply that all drainage divides move with velocity u_c. Rather, it implies that any formula describing drainage divide migration must scale as $\sqrt{DK}$, and cannot include any other terms that depend on the model parameters K, D, and U. Such a formula will also depend on factors that vary locally across the landscape (such as, for example, divide asymmetry), and which must be derived separately for specific cases; they cannot be derived by scaling considerations. These principles provide a plausibility check for theoretical predictions of drainage divide migration in landscapes that follow Eq. (1). The same general approach may also be applicable in landscapes that follow other governing equations, if one can define a characteristic velocity (which may scale differently than the example shown in Eq. 20).

3.2.2 Improving modeling efficiency

The temporal and geometric similarity of our rescaled simulations (Sect. 3.1) implies that we can explore the entire K, D, and U parameter space by rescaling a single simulation. For instance, if we are interested in how the slope–area curve depends on these three parameters, we can run one simulation with one combination of K, D, and U and plot its slope–area curve. To obtain the slope–area curve for any other combination of parameters K', D', and U', we can simply rescale slopes by the characteristic gradient and drainage areas by the characteristic area, i.e., we can multiply slopes by $\left(U'/\sqrt{D'K'}\right)/\left(U/\sqrt{DK}\right)$ and drainage areas by $(D'/K')/(D/K)$ (see Eqs. 11 and 15). The resulting rescaled drainage areas and slopes will be exactly equal to the drainage areas and slopes of a simulation with parameters K', D', and U', and with domain size, resolution, and initial conditions that are rescaled as described in Sect. 3.1.

Exploring a parameter space by rescaling can be orders of magnitude more efficient than running multiple simulations for multiple parameter combinations. For example, consider a numerical experiment exploring 10 values for each of the three parameters K, D, and U, in all possible combinations. Exploring this parameter space by brute force would require 1000 simulations, versus just one simulation with the rescaling approach.

Inferring the results of a simulation by rescaling the results of another simulation assumes that the sizes of the simulation domains are equal in units of characteristic length l_c (i.e., domain sizes are rescaled by l_c); this may often be physically unrealistic. For example, if the simulation domain represents an island, increasing l_c (e.g., by increasing the diffusion coefficient D) will make the island less dissected (i.e., will increase the spacing of valleys), but will not make the island bigger as would be required to keep the domain size constant in units of l_c. Consequently, the original island will look rougher than the island with increased l_c, and the two islands, overall, will not be geometrically similar, even in a statistical sense. Locally, however, features that are much smaller than both islands, and sufficiently far from the coastlines, will be insensitive to whether the coastlines are rescaled or not; thus, these features may be statistically similar (even if their exact spatial patterns differ). Therefore, our rescaling approach may give us insight into how model parameters control the behavior of sufficiently small features, regardless of whether domain rescaling is assumed.

However, if we vary the model parameters such that the features of interest are no longer small with respect to the domain, then these features will be influenced by boundary effects and may not be able to express their intrinsic shapes or behaviors, i.e., the shapes or behaviors that they would have if they were small relative to the domain. On the other hand, if we vary the parameters such that features of interest are not sufficiently large with respect to the resolution, then these features may be influenced by resolution effects, because they may be insufficiently resolved. In both of these cases, we can no longer reasonably assume that we can study the features of interest with a rescaling approach.

To be able to assess which combinations of domain sizes and resolutions, and model parameters K, D, and U could result in boundary or resolution effects, one should consider domain sizes and resolutions in units of characteristic length l_c. For example, the regime transition from dominant diffusion to dominant incision occurs at length scales of the order of l_c as shown by Perron et al. (2008, 2009); see also Sect. 4.2.2 below. Thus, if we want to study this regime transition, we should vary the model parameters such that l_c remains sufficiently small compared to the domain size and sufficiently large compared to the resolution. How small is sufficiently small (and, likewise, how large is sufficiently large) will not be known a priori. However, if, within a range of values of l_c, a feature's properties and behavior scale according to l_c, h_c, and t_c as

described in Sect. 3.2.1 (e.g., if drainage divide migration velocity follows Eq. (20)), then we can infer a posteriori that it can be studied with a rescaling approach over that range of l_c.

In general, one should consider all the specifications of simulations not only in units of meters and years, but also in units of l_c, h_c, and t_c (e.g., amplitudes of initial conditions in units of h_c, rates of elevation change in units of $h_c\,t_c^{-1}$, etc.). Likewise, we recommend converting simulation results into units of l_c, h_c, and t_c (e.g., drainage densities in units of l_c^{-1}, drainage areas of valley heads in units of l_c^2, response times in units of t_c, etc.). This can be helpful in comparing seemingly disparate model results, and identifying which metrics and features can be studied with a rescaling approach.

4 Interpretations of the characteristic scales

The values of the characteristic scales depend on the relative magnitudes of the model parameters K, D, and U, and thus on the relative strengths of incision, diffusion, and uplift. In the present section, we show that the characteristic scales can link the relative strengths of these processes with topographic properties of the landscape. Thus they can aid the study of process competition and regime transitions.

4.1 Height scales

4.1.1 Using height scales to quantify the vertical influence of incision, diffusion, and uplift

The incision, diffusion, and uplift terms of the governing equation (Eq. 1) give the rates of change of elevation due to the respective processes. We can scale these rates using the characteristic time t_c along with the characteristic height h_c and two additional height scales that we introduce in this section.

The definitions of the characteristic height ($h_c = U/K$; Eq. 4) and the characteristic time ($t_c = 1/K$; Eq. 5) show that h_c is the elevation uplifted per unit of t_c, i.e., $h_c = U\,t_c$. Therefore, we can view h_c as a scale that measures the contribution of uplift to elevation change per unit of t_c. We extend this notion to the incision and diffusion terms of the governing equation (Eq. 1), and define an incision height scale as the erosion due to incision per unit of t_c,

$$h_I := K\sqrt{A}|\nabla z|\ t_c = \sqrt{A}|\nabla z|\,, \tag{21}$$

and a diffusion height scale as the elevation change due to diffusion per unit of t_c,

$$h_D := D\nabla^2 z\ t_c = l_c^2\,\nabla^2 z\,. \tag{22}$$

Intuitive interpretations of h_I, h_D, and h_c are schematically illustrated in Fig. (6), which is described in more detail in the following subsection (Sect. 4.1.2). The incision height is defined as a positive quantity, but because it measures erosion we should remember that it is pointing in a downward direction. The diffusion height is negative for erosive diffusion and positive for depositional diffusion. We observe that, for the case of Eq. (1), which has drainage area and slope exponents $m = 0.5$ and $n = 1$, h_I is equal to the steepness index, defined as $k_s = A^{m/n}|\nabla z|$ (e.g., Whipple, 2001). For slope exponents $n \neq 1$, however, k_s and h_I are not equal; k_s is proportional to $h_I^{1/n}$ (Eq. A19).

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In geometrically similar landscapes, the incision and diffusion heights of corresponding points will be rescaled by the characteristic height h_c in the same way as all other variables with dimensions of height (e.g., elevations, reliefs, etc.). Specifically, assume that two geometrically similar landscapes have parameters K, D, and U, and K', D', and U' (and thus have characteristic scales l_c, h_c, and t_c, and l_c', h_c', and t_c'; note that primed variables refer to the second landscape, whose parameters are also primed). As we explain in Sect. 3 and Appendix B, corresponding points in these landscapes, i.e., points with coordinates such that $x'/l_c' = x/l_c$ and $y'/l_c' = y/l_c$ (Eqs. 18 a, b), will have drainage areas and slopes such that $\sqrt{A'}/l_c' = \sqrt{A}/l_c$ and $|\nabla'z'|/G_c' = |\nabla z|/G_c$. Therefore, they will have incision heights that are related according to:

$$h_I'/h_c' = \sqrt{A'}|\nabla'z'|/h_c' = (l_c'/l_c)\sqrt{A} \ (G_c'/G_c)|\nabla z|/h_c' = \sqrt{A}|\nabla z|/h_c = h_I/h_c \ . \tag{23 a}$$

Likewise, one can show that they will have diffusion heights that are related according to:

$$h_D'/h_c' = h_D/h_c \ . \tag{23 b}$$

Equations (23 a, b) are examples of the ability of a characteristic scale to rescale variables with which it has the same dimensions – in this case the characteristic height h_c rescales h_I and h_D even though they are not physical heights (they just have dimensions of height).

Note that Eq. (23 a) shows that, if we define a dimensionless incision height h_I^* as the ratio of the incision height h_I to the characteristic height h_c, then it will be equal to the dimensionless incision terms of Eqs. (16) and (17), i.e.,

$$h_I^* := h_I/h_c = \left(\sqrt{A}|\nabla z|\right)/\left(\sqrt{A_c} \ G_c\right) = \sqrt{A^*}|\nabla^*z^*| \ . \tag{24 a}$$

Likewise, one can show that an analogously defined dimensionless diffusion height h_D^* will be equal to the dimensionless diffusion terms of Eqs. (16) and (17), i.e.,

$$h_D^* := h_D/h_c = \nabla^2 z/\kappa_c = \nabla^{*2}z^* \ . \tag{24 b}$$

Equations (24 a, b) highlight that the three terms of the dimensionless Eqs. (16) and (17) quantify the relative contributions of incision, diffusion, and uplift to elevation change.

4.1.2 Properties of the incision, diffusion, and characteristic height scales h_I, h_D, and h_c

We can express the governing equation Eq. (1) in terms of the incision, diffusion, and characteristic height scales h_I, h_D, and h_c if we multiply it by the characteristic time t_c:

$$(\partial z/\partial t) \ t_c = -h_I + h_D + h_c \ . \tag{25}$$

In steady state ($\partial z/\partial t = 0$), Eq. (25) yields

$$0 = -h_I + h_D + h_c \ , \quad \text{for} \ \ \partial z/\partial t = 0 \ , \tag{26}$$

which we can manipulate in various ways to reveal useful properties of the three height scales h_I, h_D, and h_c.

First, we focus on points that have zero curvature ($\nabla^2 z = 0$). At these points, the net effect of diffusion on elevation is zero; thus, incision and uplift must be in balance with each other in steady state. Setting $h_D = 0$ in Eq. (26), we can mathematically express the incision–uplift balance at these points in two equivalent ways:

$$h_I = h_c \ \Leftrightarrow \ \sqrt{A}|\nabla z| = U/K \ , \quad \text{for} \ \partial z/\partial t = 0 \ , \quad \nabla^2 z = 0 \ . \tag{27}$$

These expressions show that the characteristic height h_c determines the steady-state value of the incision height h_I at points with zero curvature so that incision and uplift are in balance with each other (since diffusion has a zero net contribution to elevation change at these points). Points with zero curvature represent a regime transition between concave-down hillslopes, characterized by net erosion by diffusive transport, and concave-up valleys, characterized by net deposition by diffusive

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transport (e.g., Howard, 1994). Thus, a notable implication of Eq. (27) is that points with $h_I = h_c$ will map out this important topography- and process-related regime transition in steady state. Equation (27) is reminiscent of a bedrock river steady-state slope–area relation, and we discuss the similarities and differences between them in Sect. 4.1.3.

5   Second, we focus on drainage divides, where the drainage area A is zero and there is no incision. At drainage divides, diffusion and uplift must be in balance with each other in steady state. Setting $h_I = 0$ in Eq. (26), we express the diffusion–uplift balance on drainage divides as $h_D = -h_c$. Substituting the definitions of h_D and the characteristic curvature κ_c, and rearranging, yields the steady-state value of drainage divide curvature (e.g., Roering et al., 2007; Perron et al., 2009):

$$\nabla^2 z = -h_c/l_c^2 = -\kappa_c = -U/D \, , \quad \text{for} \ \partial z/\partial t = 0 \, , \quad A = 0 \, . \tag{28}$$

This relation shows that we can view the characteristic height h_c and length l_c as two distinct components (one vertical, the
10 other horizontal) that jointly determine the steady-state curvature of drainage divides so that diffusion and uplift are in balance.

Equations (27) and (28) refer to special points where incision or diffusion is zero. These are special cases of a general steady-state property of h_c that is valid at all points; rearranging Eq. (26) yields

$$h_c = h_I - h_D \, , \quad \text{for} \ \partial z/\partial t = 0 \, , \tag{29}$$

15 which shows that the difference $h_I - h_D$ is constant and equal to h_c across steady-state landscapes.

Figure 6 schematically illustrates how the incision, diffusion, and characteristic height scales h_I, h_D, and h_c vary along a steady-state profile. In Fig. 6 b, the green line shows a steady-state profile that traces a flow path from the drainage divide to a point in a valley. Subtracting h_I from the elevations along the profile, or adding h_D or h_c to them, yield three gray lines, one
20 solid, one dashed, and one dotted, that respectively show the individual contributions of incision, diffusion, and uplift to elevation change per unit of t_c. (These contributions are equivalent to how the profile would change if only incision, diffusion, or uplift operated on it, at their equilibrium rates, for one unit of t_c.) The three contributions must sum to zero at all points along this equilibrium profile. Whereas Fig. 6 b shows elevations and elevation changes, Fig. 6 a shows the values of h_I, h_D, and h_c along the profile, using black lines that have the same shapes as the corresponding gray lines of Fig. 6 b. Figure
25 6 a schematically illustrates the relationships described by Eqs. (27), (28), and (29). Specifically, at the divide (point P_1), h_I is zero and h_c and h_D are equal and opposite; at the point of zero curvature (point P_2; also shown magnified in Fig. 6 b), h_D is zero and h_I equals h_c; and over the entire profile, $h_c = h_I - h_D$ (the spacing between the dashed and solid black lines is constant and equal to h_c).

30 Substituting the definitions of h_I and h_D (Eqs. 21, 22) into Eq. (29) yields

$$h_c = \sqrt{A}|\nabla z| - l_c^2 \nabla^2 z \, , \quad \text{for} \ \partial z/\partial t = 0 \, , \tag{30}$$

which shows that the constant difference $h_I - h_D$ implies that, in steady state, drainage areas A, slopes $|\nabla z|$, and curvatures $\nabla^2 z$ are constrained by a relationship that is constant across the landscape and is parameterized by h_c (along with l_c). In this sense, we can interpret the characteristic height h_c as a parameter that constrains the steady-state values of the drainage area A, slope $|\nabla z|$, and curvature $\nabla^2 z$ across the landscape so that incision, diffusion, and uplift are in balance.
35

For drainage area and slope exponents m and n such that $2m \neq n$ (Eq. A1), h_c depends on all three parameters K, D, and U (Eq. A9). Therefore, we should not interpret the characteristic height h_c as a scale that expresses the relative strength of uplift

versus incision (as the definition $h_c = U/K$ (Eq. 4) may seem to suggest), but rather interpret it as expressing the relative strengths of all three processes. ~~This is in line with~~ Thus, the aforementioned interpretation that h_c constrains steady-state topography so that all three processes are in balance is in line with the definition of h_c for generic exponents m and n.~~,~~ ~~including diffusion, which may seem odd given that Eq. (4) defines h_c as independent of the diffusion coefficient D. But as~~ ~~the generic definition of h_c (Eq. A9) shows, D drops out only when $2m = n$, which is true for the governing equation~~ ~~Eq. (1).~~

Given that Eq. (30) can be rewritten as

$$\sqrt{A}|\nabla z| = l_c^2 \, \nabla^2 z + h_c \,, \quad \text{for} \quad \partial z / \partial t = 0 \,, \tag{31}$$

if we plot the product $\sqrt{A}|\nabla z|$ versus the curvature $\nabla^2 z$ we can graphically illustrate how h_c and l_c constrain A, $|\nabla z|$, and $\nabla^2 z$ in a steady-state landscape. Additionally, we can graphically illustrate the special cases on points with zero curvature and on drainage divides described by Eqs. (27) and (28), respectively. We show such a plot in Fig. 7 using data from the simulated equilibrium landscape B (shown in Figs. 1 and 2). Figure 7 shows that $\sqrt{A}|\nabla z|$ versus $\nabla^2 z$ plot on a straight line, as demanded by Eq. (31). In Fig. 7, the vertical dashed lines show the values $\nabla^2 z = 0$ and $\nabla^2 z = \kappa_c$; the horizontal dashed lines show the values $\sqrt{A}|\nabla z| = h_c$ and $\sqrt{A}|\nabla z| = 2h_c$. Therefore, these dashed lines illustrate that the straight line on which the data points plot has an intercept equal to the characteristic height h_c, and a slope equal to h_c/κ_c, i.e., equal to l_c^2, the square of the characteristic length, as demanded by Eq. (31).

4.1.3 Quantitative predictions

Equations (30) and (31), the relationships that constrain steady state topography, are testable predictions. They imply that we can test whether the governing equation Eq. (1) describes a given, presumably steady-state, real-world landscape by plotting estimates of the product $\sqrt{A}|\nabla z|$ versus estimates of the curvature $\nabla^2 z$ from across this landscape. These estimates should plot on a straight line, as shown in Fig. 7 for our simulated landscape. (Note that our analysis only shows that Eqs. (30) and (31) must hold for a steady-state landscape governed by Eq. (1); we have not shown that a landscape that conforms to Eqs. (30) and (31) is necessarily governed by Eq. (1), or in steady state. Although it seems unlikely that data from a non-steady-state landscape that follows different geomorphic transport laws would happen to plot according to Eqs. (30) and (31), this premise should be tested using numerical experiments.)

Equation (31) can be used to estimate model parameters. Specifically, we can estimate l_c^2 (i.e., the D/K ratio) from the slope of plots of the product $\sqrt{A}|\nabla z|$ versus the curvature $\nabla^2 z$, and estimate h_c (i.e., the U/K ratio) from the intercept of these plots. Alternatively, given that $\kappa_c = h_c / l_c^2$ (Eq. 13), we can rewrite Eq. (31) as

$$\sqrt{A}|\nabla z| = \frac{h_c}{\kappa_c} \nabla^2 z + h_c = h_c \frac{\nabla^2 z + \kappa_c}{\kappa_c} \,, \tag{32}$$

and estimate h_c as the slope of plots of the product $\sqrt{A}|\nabla z|$ versus the quantity $(\nabla^2 z + \kappa_c)/\kappa_c$. To apply Eq. (32) to real-world data we would need to estimate κ_c, e.g., as discussed above from steady-state drainage-divide curvature (e.g., Perron et al., 2009). Note that Eqs. (30)–(32) are equivalent to Eq. (5) of Perron et al. (2009), which they used to estimate l_c and which they recognized as a test for model validity. For instance, dividing Eq. (32) by κ_c yields Eq. (5) of Perron et al. (2009).

Another equation that could hypothetically be used to estimate model parameters is Eq. (27), which is reminiscent of the steepness-index formula $A^{m/n}|\nabla z| = (U/K)^{1/n}$ that has been used to estimate model parameters from steady-state profiles of bedrock rivers (e.g., Sklar and Dietrich, 1998; Whipple, 2001). Although the two equations are similar in form, they apply to different points on the landscape. Equation (27) describes points of zero curvature, where diffusive transport is negligible, whereas the steepness-index formula is applied to bedrock rivers, where curvature is clearly not zero but diffusive transport is nonetheless assumed to be zero (and thus Eq. (1) does not apply). To estimate h_c, Eq. (27) can only use the relatively few points with zero curvature, whereas Eq. (32) can use data from the whole landscape and, thus, would presumably yield more robust estimates of h_c.

4.1.4 Valley definition

Valleys have been defined as areas where the quantity $A(|\nabla z|)^2$ exceeds some threshold value (e.g., Montgomery and Dietrich, 1992; Orlandini et al., 2011; Clubb et al., 2014). This quantity is equal to $(h_I)^2$, the square of the incision height. Montgomery and Dietrich (1992) used thresholds of $A(|\nabla z|)^2$ as criteria for defining channels and valleys, concluding that channelization and valley incision are controlled by the same topographic properties. Other authors have used curvature $\nabla^2 z$ to define valleys; specifically, valleys have been defined as regions with curvature above some threshold, i.e., $\nabla^2 z \geq \kappa_{thr}$, where the threshold curvature κ_{thr} is assumed to be zero (e.g., Howard, 1994) or a small positive value (e.g., Lashermes et al., 2007; Pelletier, 2013). Here we demonstrate that these seemingly disparate criteria are closely related in steady-state landscapes that follow Eq. (1).

Equations (26) and (31) can be combined as

$$h_I = \sqrt{A}|\nabla z| = l_c^2\,\nabla^2 z + h_c, \quad \text{for } \partial z/\partial t = 0, \tag{33}$$

which shows that, in steady state, $\sqrt{A}|\nabla z|$ – the square root of Montgomery and Dietrich's valley criterion – is linearly related to topographic curvature $\nabla^2 z$. Equation (33) shows that points with $\nabla^2 z \geq \kappa_{thr}$ will be identical to points whose incision heights $h_I = \sqrt{A}|\nabla z|$ exceed a corresponding threshold value h_{thr}. Figure 8 graphically illustrates the property of h_I to define valleys. It shows that coloring the simulated equilibrium landscape B (shown in Figs. 1 and 2) using the h_I value of each pixel reveals a dendritic pattern. Given that h_I and $\nabla^2 z$ are linearly related, they have identical spatial patterns; thus, the dendritic pattern of Fig. 8 shows the valley network defined by either criterion.

Given that one is a linear function of the other, are there practical reasons to prefer $A(|\nabla z|)^2$ or $\nabla^2 z$ as a valley definition criterion? The first criterion requires computing drainage areas to each point on the landscape, which can be computationally tedious, whereas the second criterion requires estimating curvatures, which can be sensitive to topographic noise. Note, however, that thresholds of the quantity $A(|\nabla z|)^2$ correspond to curvature thresholds only if $2m = n$. In the more general case of $2m \neq n$ (Eq. A1), curvature thresholds will correspond to thresholds of the quantity $A(|\nabla z|)^{n/m}$ (Eq. A26), rather than $A(|\nabla z|)^2$ (e.g., Ijjasz-Vasquez and Bras, 1995).

4.1.5 Divide migration dynamics

Equation (33) holds only in steady state; its counterpart in transient states would be $(\partial z/\partial t)t_c + h_I = l_c^2\,\nabla^2 z + h_c$, as implied by Eq. (25). In transient states, the term $(\partial z/\partial t)t_c$ varies across the landscape. Thus the incision height h_I is not linearly related with curvature $\nabla^2 z$, but nonetheless it remains a useful quantity. For instance, it reproduces Whipple et al.'s

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(2017) finding that erosion rate differences across drainage divides can predict the direction of divide migration. In Fig. 9 we show four snapshots of the simulated evolving landscape B (shown in Figs. 3 and 4), which we colored using the h_I value of each pixel. This coloring revealed that h_I's spatial distribution followed dendritic patterns. Furthermore, it revealed that dendritic patterns across migrating drainage divides had different colors, i.e., different h_I values. Finally, it revealed that

5   drainage basins overwhelmingly tended to expand at the expense of neighbors with dendritic patterns with relatively lower h_I values. Figure 9 suggests that h_I predicts the direction of divide migration. This property characterized the entire evolution of landscape B, but was more evident during the early phases from which we chose the four snapshots of Fig. 9. For the case of Eq. (1), h_I is equal to the steepness index k_s, which was one of the metrics that Whipple et al. (2017) used to measure erosion rates; thus Fig. 9 also illustrates the use of k_s to predict divide migration dynamics.

10   ## 4.2 Length and time scales

Perron et al. (2008, 2009, 2012) expressed the competition between diffusion, which smooths landscapes, and incision, which dissects them, in terms of a Péclet number $\text{Pe} = K l^2 / D$, where l is a length scale that characterizes landscape features of interest. Their analysis implied that incision and diffusion should be equally effective when the Péclet number is roughly equal to one, and thus the length scale is roughly equal to $\sqrt{D/K}$, i.e., to the characteristic length l_c. They showed that

15   distances between equally spaced valleys scale with l_c, while drainage areas of first-order valley heads and of second- to first-order valley branching scale with l_c^2, i.e., with the characteristic area A_c. Here, we introduce a related, but different, definition of the Péclet number and explore its implications. One such implication is that we can use this Péclet number to interpret the characteristic length and time scales l_c and t_c as scales that characterize a transition between regimes of dominant diffusion and dominant incision.

20   ### 4.2.1 Quantifying the horizontal influence of incision and diffusion

To examine the properties of l_c we focus on the horizontal effects of incision, diffusion, and uplift. This approach is in line with our goal to study horizontal and vertical scaling separately, and with the assumption that the dimensions of length and height are distinct. Equation (1) explicitly defines rates of elevation change, i.e., effects of processes in the vertical direction. However, the incision and diffusion terms have additional, implicit horizontal effects. Specifically, incision advects

25   topographic perturbations, such as knickpoints, and diffusion smooths them (e.g., Whipple and Tucker, 1999; Perron et al., 2008). We can quantify the strength of these effects using time scales that characterize incision and diffusion.

The incision term of Eq. (1) has the form of a kinematic wave, and advects perturbations at a rate equal to the kinematic wave celerity $c = K\sqrt{A}$ (e.g., Whipple and Tucker, 1999). The time needed to advect perturbations over some distance l

30   gives an appropriate measure of incision's horizontal influence; thus, we define an incision time scale as:

$$t_I \coloneqq \frac{l}{K\sqrt{A}}. \tag{34}$$

A small t_I value corresponds to a strong horizontal influence of incision (the stronger the incision, the less time is needed for advection over a given distance l).

The diffusion term of Eq. (1) smooths elevation differences by redistributing them over an expanding region of neighboring

35   points. For example, an elevation difference that is initially concentrated at a single point will evolve as a Gaussian function,

centered around this point and with a standard deviation that grows proportionally to $\sqrt{Dt}$. In general, all elevation perturbations will spread proportionally to $\sqrt{Dt}$ regardless of their initial shape, because the diffusion term of Eq. (1) is linear and thus the superposition principle applies (e.g., Balluffi et al., 2005). Thus, the quantity $\sqrt{Dt}$ is a length scale that characterizes how far diffusion can spread elevation perturbations during some time t. Consequently, the diffusion time scale

5   (e.g., Perron et al., 2008, 2012)

$$t_D := \frac{l^2}{D},\tag{35}$$

which is equal to the time needed for the quantity $\sqrt{Dt}$ to reach some value l, is an appropriate measure of diffusion's horizontal influence. Analogously to t_I, a small t_D value corresponds to a strong horizontal influence of diffusion.

Following Perron et al. (2008, 2009, 2012), we quantify the relative horizontal influence of incision versus diffusion across a

10   landscape by the ratio of the diffusion time t_D to the incision time t_I. Using the definitions in Eqs. (34) and (35) leads to the following definition of the Péclet number Pe:

$$\text{Pe} := t_D/t_I = \frac{K\sqrt{A}\,l}{D} = \frac{\sqrt{A}\,l}{l_c^2}.\tag{36}$$

Given that small t_D or t_I values correspond to strong influences of the respective process, small Pe values correspond to hillslopes, where diffusion is horizontally dominant, and large Pe values correspond to valleys, where incision is horizontally dominant.

15

The transition between the regimes of horizontally dominant diffusion and incision can be assumed to occur where the incision and diffusion time scales are roughly equal, i.e., where $t_I \approx t_D$ or, equivalently, where $\text{Pe} \approx 1$. Substituting this value into the definition of the Péclet number (Eq. 36) yields

$$\text{Pe} \approx 1 \quad \Leftrightarrow \quad \sqrt{A}\,l \approx l_c^2.\tag{37}$$

As mentioned above, Perron et al.'s (2008) Péclet number is equal to one for a length scale equal to l_c. In contrast, Eq. (37)

20   shows that our Péclet number is equal to one not for specific values of the length scale l or the drainage area A, but rather for all combinations of values for which $\sqrt{A}\,l$ is equal to l_c^2. Note that $l_c^2 = \sqrt{A_c}\,l_c$; thus, if both $l \approx l_c$ and $A \approx A_c$, then $\text{Pe} \approx 1$. Furthermore, note that for $l \approx l_c$ and $A \approx A_c$ the incision and diffusion times are both roughly equal to the characteristic time, i.e., $t_I \approx t_c$ and $t_D \approx t_c$. Thus, the characteristic length, area, and time scales can be interpreted as characterizing the regime transition from dominant diffusion to dominant incision.

25   **4.2.2   Spatial distribution of the Péclet number across a landscape**

To calculate values of the Péclet number Pe across a landscape we must specify what the length scale l is. In this study, we define l as the maximum distance along flow paths from a point to the drainage divide, and refer to it as the point's flow path length. Because the incision term has the form of a kinematic wave (and kinematic waves propagate only in the direction of the gradient; Lighthill and Whitham, 1955), perturbations are advected only in the uphill direction along flow paths. Thus, it

30   is reasonable to quantify advection's influence using a length scale that is measured along flow paths and points in the uphill direction; the flow path length is such a length scale. Because the diffusion term spreads out elevation differences in all directions (uphill and downhill along flow paths, and laterally across flow paths), its influence could be quantified using several length scales, including the flow path length. Because the flow path length is the natural length scale for the incision

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term, and one of several possible length scales for the diffusion term, it is a reasonable length scale for calculating the Péclet number, which measures the relative influence of advection versus diffusion.

Figure 10 shows how the Péclet number Pe varies across a landscape. We calculated Pe according to Eq. (36), assuming that l is the flow path length and using data from the simulated equilibrium landscape B (as shown in Figs. 1, 2, and 8). Each pixel in Fig. 10 is colored by $\log_{10}$ Pe. We used the logarithm to visualize values of Pe that are both much smaller and much larger than 1. Figure 10 shows that the highest values of Pe follow the dendritic valley network, similar to the highest values of the incision height h_I (see Fig. 8). Diffusion and incision are horizontally dominant on hillslopes and in valleys, respectively. Therefore, the dendritic patterns in Fig. 10 suggest that our Péclet number (defined by Eq. (36) and calculated using the flow path length) is a reasonable measure of the relative horizontal influence of incision versus diffusion.

Just as the incision and diffusion heights h_I and h_D can be rescaled by the characteristic height h_c (Eqs. 23 a, b), one can show that, at corresponding points in geometrically similar landscapes, the incision and diffusion times t_I and t_D can be rescaled by the characteristic time t_c, i.e.,

$$t_I'/t_c' = t_I/t_c , \qquad\qquad t_D'/t_c' = t_D/t_c . \qquad\qquad\qquad\qquad (38\ \text{a, b})$$

Furthermore, one can show that, at corresponding points in geometrically similar landscapes, Péclet numbers are equal, i.e.,

$$\text{Pe}' = \sqrt{A'}\, l'/l_c'^2 = \sqrt{A}\, l/l_c^2 = \text{Pe} . \qquad\qquad\qquad\qquad (38\ \text{c})$$

In other words, if we would plot maps of Pe using data from the simulated landscapes A and C, then we would obtain rescaled copies of the map of Pe derived from landscape B (as seen in Fig. 10).

4.2.~~3~~4 Implications of the new Péclet number definition

In this subsection, we discuss the differences between the Péclet number defined in this study and the Péclet number defined by Perron et al. (2008, 2009, 2012). Our definition (Eq. 36) includes both the drainage area A and the length scale l, whereas theirs included only l. Specifically, our Péclet number is defined as $\text{Pe} = \sqrt{A}\, l/l_c^2$, whereas theirs was defined as $\text{Pe} = Kl^2/D = l^2/l_c^2$ (for drainage area and slope exponents $m=0.5$ and $n=1$). To introduce their Péclet number, Perron et al. (2008) defined an incision-term celerity that implicitly assumed that $l = \sqrt{A}$, and in this way they substituted l^2 for $\sqrt{A}\, l$. In contrast, we defined the incision and diffusion time scales, and the Péclet number (Eqs. 34–36), using an abstract length scale l, and we calculated values of the Péclet number assuming that l is the flow path length.

To determine how our Péclet number scales with the flow path length l or the drainage area A, we first need to explore the scaling relationship between l and A. The flow path length l scales as a power law of the drainage area A, i.e., as A^p, with an exponent p that depends on how convergent or divergent the topography is. (Here, we refer to horizontal topographic convergence or divergence, as measured, for example, by the contour curvature.) In convergent contributing areas, the scaling exponent is $p = 0.5$ (i.e., l scales as $\sqrt{A}$; e.g., Montgomery and Dietrich, 1992), in planar contributing areas, it is $p = 1$ (i.e., l scales linearly with A; e.g., Pelletier, 2010), and in divergent topography, it is $p > 1$. In general, the topography of contributing areas is not purely convergent, planar, or divergent; it is a mixture of these three types. In such cases, the scaling exponent p will be somewhere between the values of the three types of topography, depending on how they are mixed. At large scales, l will generally scale as $\sqrt{A}$ (e.g., Mueller, 1972), which corresponds to convergent topography.

Figure 11 schematically illustrates how the scaling relationship between flow path length l and drainage area A depends on the topography of contributing areas. It shows a point inside a valley (P_1) that has convergent contributing topography, a point on a hillslope (P_2) that has planar contributing topography, and a point on an interfluve (P_3) that has divergent contributing topography. Figure 11 shows that, even though the three points have equal flow path lengths, they have

5   different drainage areas, and those drainage areas scale differently with flow path length. Additionally, Fig. 11 illustrates how contributing areas can have mixed topographies; for example, the contributing area of P_1 becomes less convergent and more planar near the ridge.

Given that the Péclet number Pe is proportional to $\sqrt{A}\, l$ (Eq. 36) and that l scales as A^p, the Péclet number will scale as

10   $l^{1+1/(2p)}$ or $A^{p+1/2}$. The values of the exponents $1 + 1/(2p)$ and $p + 1/2$ are determined by the value of p, which depends on topography as described above. Therefore, in convergent topography, and at large scales, Pe will scale as l^2 or A. At small scales in non-convergent or mixed topography, Pe will scale with l raised to a power less than 2, or with A raised to a power greater than 1.

15   Perron et al.'s (2008, 2009, 2012) Péclet number scales as l^2. To calculate Pe across real-world landscapes, Perron et al. (2012) defined the length scale l as the length of basins and calculated it from drainage area data according to $l = \sqrt{3A}$; in this case their Péclet number scales as A. Consequently, in convergent topography, and at large scales, both their Péclet number and ours scale as l^2 or as A. In contrast, at small scales in non-convergent or mixed topography, the two Péclet numbers scale differently from each other with l and A.

20   **5   Summary and conclusions**

In this study, we perform dimensional analysis on an LEM that includes terms for stream-power incision, linear diffusion, and uplift (Eq. (1); e.g., Howard, 1994; Dietrich et al., 2003). The governing equation that we analyze in the main text (Eq. 1) includes the relatively simple incision term $K\sqrt{A}|\nabla z|$, which is a special case of the more general incision law $KA^m(|\nabla z|)^n$ (Eq. A1). As we demonstrate in Appendix A, results obtained from dimensional analysis of the LEM with the

25   general incision law (Eq. A1) are equivalent to results obtained from the LEM with the simple incision law (Eq. 1), but the latter have much simpler forms and, thus, are more suitable for presentation in the main text.

Our dimensional analysis is based on two key premises. First, we assume that the dimensions of length and height are conceptually distinct. Second, we use only intrinsic characteristic scales, i.e., scales that depend only on the parameters of

30   the model (the incision coefficient K, the diffusion coefficient D, and the uplift rate U), not on sizes of the domain or of landscape features. We use the characteristic scale l_c (Eq. 3) previously defined by Perron et al. (2008), and we introduce new characteristic height and time scales h_c and t_c. (Eqs. 4 and 5). The use of these three characteristic scales allows us to obtain three main results.

35   First, rescaling the governing equation (Eq. 1) by l_c, h_c, and t_c yields a dimensionless form (Eq. 16) that includes only variables and no parameters. Because it has no parameters that can be adjusted, this dimensionless equation has only one solution for any given set of (dimensionless) boundary and initial conditions. This result means that landscapes that are

rescaled horizontally by l_c (Eqs. 18 a, b) and whose initial conditions are rescaled vertically by h_c (Eq. 18 c) will follow temporally and geometrically similar evolutions, i.e., if we compare these landscapes at times that are rescaled by t_c (Eq. 18 d), then they will be copies of each other (rescaled horizontally by l_c and vertically by h_c). We demonstrate the temporal and geometric similarity of rescaled landscapes theoretically (Appendix B) and numerically (Sect. 3.1, Figs. 1–5).

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Second, l_c, h_c, and t_c can be combined to define other characteristic scales (e.g., characteristic velocities, slopes, and curvatures), which rescale variables whose dimensions combine L, H, and T in the same way. Based on these definitions of characteristic scales, we can straightforwardly deduce scaling relations between any landscape metric and the model parameters K, D, and U, because all of our scales are defined to depend only on parameters. As an example, we present a

10 characteristic horizontal velocity that must rescale the migration velocity of drainage divides (Eq. 20).

The temporal and geometric similarity of rescaled landscapes implies that we can explore all combinations of the model parameters K, D, and U by simulating a single combination of parameters for any given dimensionless domain size, resolution, and initial conditions. We can then simply rescale the results of this simulation to obtain any results for any other

15 combinations of parameters, which is significantly more efficient than running multiple simulations for multiple parameter combinations.

Such a modeling approach assumes that simulation domains are rescaled, but this is not always physically realistic (e.g., if a domain represents an island, changing model parameters may change the sizes of ridges and valleys on the island, but not the

20 size of the island itself). Nonetheless, as we explain in Sect. 3.2.2, landscape features that are sufficiently small with respect to the domain size may locally remain statistically similar, even if the landscapes globally are not similar. Therefore, the rescaling approach may offer insights into how such features depend on model parameters, even if the domain is not rescaled.

25 However, if landscapes are not rescaled and model parameters are varied too widely, then boundary or resolution effects may arise. These boundary and resolution effects will be minimized if the domain size is much larger than l_c and the resolution is much smaller than l_c. More generally, we recommend performing model simulations in units of the characteristic scales (i.e., in dimensionless terms) instead of in conventional dimensional units. This should be helpful in comparing disparate model results and identifying which features can be rescaled.

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Third, l_c, h_c, and t_c can be interpreted as expressing the competition between incision, diffusion, and uplift, and as linking the relative strengths of these processes to topographic properties of the landscape (Sects. 4.1, 4.2). This interpretation is facilitated by introducing process-specific height and time scales h_I, h_D, t_I, and t_D (Eqs. 21, 22, 34, and 35) that measure the vertical and horizontal influences of incision and diffusion. The incision and diffusion heights quantify the contribution of

35 these processes to the total elevation change per unit t_c. The incision and diffusion time scales quantify how long it takes for these processes to propagate elevation perturbations over some horizontal length scale (i.e., the stronger a process, the smaller its time scale).

In steady-state landscapes, the characteristic height h_c is everywhere equal to the difference between the incision and

40 diffusion height scales, i.e., $h_c = h_I - h_D$ (Eq. 29), and in this way h_c expresses the balance between incision, diffusion, and

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uplift. Given that we define h_I as a function of drainage area and slope, and h_D as a function of curvature, $h_c = h_I - h_D$ expresses how steady-state topography is linked to the balance between incision, diffusion, and uplift (Eq. 30, Figs. 6, 7). Equations (29) and (30) show that h_c expresses the balance between all three processes, not only between incision and uplift as the definition $h_c = U/K$ (Eq. 4) seemingly suggests. This can also be seen by the fact that, for the case of the LEM with the general incision law (Eq. A1), the definition of h_c includes all three parameters (Eq. A9).

Equation (30) is a testable prediction that can discriminate between landscapes that are in steady state and follow the governing equation (Eq. 1), and those that do not. Furthermore, it can be rearranged to estimate l_c and h_c, i.e., the parameter ratios D/K and U/K. Finally, it implies that the incision height h_I is linearly related to curvature (Eq. 33). Thus, the spatial distribution of h_I follows the valley network (Fig. 8), and h_I can be used as a proxy for curvature to define the hillslope–valley transition.

The characteristic length and time scales l_c and t_c characterize the regime transition from diffusion-dominated to incision-dominated propagation of topographic perturbations. Following Perron et al. (2008, 2009, 2012), we quantify the relative influence of incision versus diffusion by the ratio of the diffusion timescale t_D to the incision time scale t_I. This leads to a new definition of the Péclet number (specifically, $\mathrm{Pe} = \sqrt{A}\, l/l_c^2$, where l is the flow path length to the divide; Eq. 36). The spatial distribution of this Péclet number follows the valley network (Fig. 10), with small Pe values corresponding to diffusion-dominated features (ridges and hillslopes) and large Pe values corresponding to incision-dominated features (valleys). The transition occurs where Pe is roughly equal to one (i.e., where t_I and t_D are roughly equal). This condition is satisfied by combinations of length scale l and drainage area A for which $\sqrt{A}\, l \approx l_c^2$ (Eq. 37). One such combination is $l \approx l_c$ and $A \approx l_c^2$, in which case t_I and t_D are both also roughly equal to t_c.

Our definition of the Péclet number differs from Perron et al.'s (2008, 2009, 2012) in that ours includes both the length scale l and the drainage area A, while theirs includes only l. Perron et al.'s definition implicitly assumes that $l = \sqrt{A}$. We assume that l is the flow path length, i.e., the maximum distance along the flow paths from a point to the divide. In this way our Pe expresses how topographic convergence and divergence control the relative importance of incision versus diffusion across the landscape (Sect. 4.2.3, Fig. 11). In convergent topography, the two Pe definitions (ours and Perron et al.'s) both scale as l^2. However, in planar or divergent topographies, Perron et al.'s Pe still scales as l^2 whereas ours scales as l^p with $p \leq 3/2$.

To summarize, l_c, h_c, and t_c lead to a dimensionless form of the governing equation that significantly simplifies the model. Additionally, they can be combined to rescale any variables with dimensions in L, H, and T. Finally, and they express process competitions and link them to topographic properties. The ability of l_c, h_c, and t_c to perform all of these tasks is a direct consequence of the two key premises underlying our dimensional analysis (characteristic scales are intrinsic, and horizontal lengths and vertical heights are dimensionally distinct). Our analysis suggests that l_c, h_c, and t_c, as a group, are fundamental properties of landscapes that follow equations such as Eqs. (1) or (A1). This in turn suggests that l_c, h_c, and t_c, or their combinations, may explain additional properties of ridge and valley topography, beyond those that we briefly mention here for demonstration purposes. Therefore, it may be illuminating to estimate l_c, h_c, and t_c in future landscape studies, whether in the field, in laboratory experiments, or in computer simulations.

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Appendix A: Dimensional analysis of LEM with generic slope, area exponents *m*, *n*

A1 Generic governing equation

We perform dimensional analysis of an LEM with an incision term with generic drainage area and slope exponents *m* and *n* that follows the governing equation

$$\frac{\partial z}{\partial t} = -KA^m(|\nabla z|)^n + D\nabla^2 z + U \ . \tag{A1}$$

5 A2 Dimensions and characteristic scales

The variables and parameters of Eq. (A1) have the following dimensions. Coordinates of points (x, y) have dimensions of L, elevation z has dimensions of H, and time t has dimensions of T. Therefore, using square brackets to denote the dimensions of quantities, we obtain $[\partial z/\partial t] = $ H T^{-1}, $[A] = $ L^2, $[|\nabla z|] = $ H L^{-1}, $[\nabla^2 z] = $ H L^{-2}. Consequently, the coefficient of incision K has dimensions of

$$[K] = \frac{[\partial z/\partial t]}{[A^m(|\nabla z|)^n]} = \frac{\text{H T}^{-1}}{\text{L}^{2m}\,\text{H}^n\,\text{L}^{-n}} = \text{L}^{n-2m}\,\text{H}^{1-n}\,\text{T}^{-1} \ . \tag{A2}$$

10 Likewise, the coefficient of diffusion and the uplift rate have dimensions of

$$[D] = \text{L}^2\,\text{T}^{-1} \ , \tag{A3}$$

$$[U] = \text{H T}^{-1} \ . \tag{A4}$$

All terms of Eq. (A1) have dimensions that can be expressed in terms of L, H, and T. Therefore, we can non-dimensionalize Eq. (A1) using a characteristic length l_c, a characteristic height h_c, a characteristic time t_c, and combinations thereof. We wish to define l_c, h_c, and t_c as functions of only the parameters K, D, and U, and the exponents m and n. We can utilize Eqs. (A2)–

15 (A4) to derive these functions. Below, we present how the definition of l_c is obtained. The definitions of h_c and t_c can be obtained analogously.

We assume that l_c will be the product of three power laws of the three parameters K, D, and U, i.e., $l_c = K^k D^d U^u$, where k, d, and u are unknown exponents that we seek to determine. The two sides of this definition of l_c must have the same

20 dimensions, i.e., $[l_c] = [K^k D^d U^u]$, which implies, in combination with Eqs. (A2)–(A4), that

$$\text{L} = [K]^k[D]^d[U]^u = (\text{L}^{n-2m}\,\text{H}^{1-n}\,\text{T}^{-1})^k(\text{L}^2\,\text{T}^{-1})^d(\text{H T}^{-1})^u = \text{L}^{(n-2m)k+2d}\,\text{H}^{(1-n)k+u}\,\text{T}^{-k-d-u} \ . \tag{A5}$$

We can find the exponents k, d, and u by requiring that in the right-hand side of Eq. (A5), the exponent of L is equal to one and the exponents of H and T are equal to zero, i.e., by solving the system

$$\begin{cases} (n-2m)k & +2d & = 1 \\ (1-n)k & +u & = 0 \ . \\ -k & -d & -u & = 0 \end{cases} \tag{A6}$$

The solution of this system is

$$k = \frac{-1}{n+2m} \ , \quad d = \frac{n}{n+2m} \ , \quad u = \frac{1-n}{n+2m} \ , \tag{A7}$$

25 i.e., $\text{L} = (\,[K]^{-1}[D]^n[U]^{1-n}\,)^{1/(n+2m)}$. Therefore, the characteristic length can be defined as:

$$l_c = (K^{-1}D^n U^{1-n})^{1/(n+2m)} . \tag{A8}$$

Following the same procedure we find definitions of the characteristic height and time scales:

$$h_c = (K^{-2}D^{n-2m}U^{2-n+2m})^{1/(n+2m)} , \tag{A9}$$

$$t_c = (K^{-2}D^{n-2m}U^{2-2n})^{1/(n+2m)} . \tag{A10}$$

For $m = 0.5$ and $n = 1$, Eqs. (A8)–(A10) yield the corresponding Eqs. (3)–(5) in the main text.

Note the exponents of the parameters K, D, and U in the definitions of l_c, h_c, and t_c (Eqs. A8–A10). The following
5   observations can be made:

- For $n = 1$, the definitions of l_c and t_c are simplified, because U drops out.
- For $n = 2m$, the definitions of h_c and t_c are simplified, because D drops out.
- D would drop out of l_c's definition if $n = 0$ and U would drop out of h_c's definition if $n = 2 + 2m$. Such exponents would not yield physically meaningful stream-power or shear-stress incision laws.
10   - K never drops out of any of the three definitions.
- Therefore, eliminating U from l_c's and t_c's definitions (for $n = 1$), or eliminating D from h_c's and t_c's definitions (for $n = 2m$) are the only simplifications of the definitions of characteristic scales that correspond to meaningful stream-power or shear-stress incision laws.
- Equation (1) satisfies both $n = 1$ and $n = 2m$; therefore, it results in the simplest set of characteristic scales that are
15   physically meaningful.

Based on the definitions of l_c, h_c, and t_c we define a characteristic area

$$A_c = l_c^2 = (K^{-2}D^{2n}U^{2-2n})^{1/(n+2m)} , \tag{A11}$$

a characteristic gradient

$$G_c = h_c/l_c = (K^{-1}D^{-2m}U^{1+2m})^{1/(n+2m)} , \tag{A12}$$

and a characteristic curvature

$$\kappa_c = G_c/l_c = h_c/l_c^2 = U/D . \tag{A13}$$
20   Note that the uplift rate can be viewed as a characteristic rate of elevation change, because

$$h_c/t_c = U . \tag{A14}$$

We define a characteristic horizontal velocity as

$$u_c = l_c/t_c = (K^1 D^{2m} U^{-1+n})^{1/(n+2m)} . \tag{A15}$$

We have shown that the characteristic scales of the governing equation in the main text, Eq. (1), are consistent with the
25   characteristic scales of Eq. (A1). Below we confirm that all results and interpretations that refer to Eq. (1) and were presented in the main text are also consistent with Eq. (A1).

A3   Dimensionless governing equation

We define dimensionless variables according to Eqs. (6, 8, 10, 12, and 14) (using the characteristic scales defined by Eqs. A8–A13) and substitute into Eq. (A1):

$$\frac{\partial z}{\partial t} = -KA^m(|\nabla z|)^n + D\nabla^2 z + U \iff$$

$$\frac{h_c \partial z^*}{l_c \partial t^*} = -K\, A_c^m A^{*m}\, G_c{}^n(|\nabla^* z^*|)^n + D\,\kappa_c \nabla^{*2} z^* + U \iff \dots \iff U\frac{\partial z^*}{\partial t^*} = U\big(-A^{*m}(|\nabla^* z^*|)^n + \nabla^{*2} z^* + 1\big) \iff$$

$$\frac{\partial z^*}{\partial t^*} = -A^{*m}(|\nabla^* z^*|)^n + \nabla^{*2} z^* + 1 \,. \tag{A16}$$

A4 Height scales

We define an incision height as the erosion due to incision during one unit of t_c

$$h_I = KA^m(|\nabla z|)^n\, t_c = A^m(|\nabla z|)^n\, (K^{n+2m-2}D^{n-2m}U^{2-2n})^{1/(n+2m)} \,. \tag{A17}$$

Note that the dimensionless incision height is:

$$\frac{h_I}{h_c} = \frac{KA^m(|\nabla z|)^n\, t_c}{h_c} = \frac{A^m(|\nabla z|)^n}{U/K} \,. \tag{A18}$$

5   The incision height h_I is related with the steepness index k_s according to:

$$k_s = A^{m/n}|\nabla z| = \left(\frac{h_I}{K\, t_c}\right)^{1/n} \,. \tag{A19}$$

We define a diffusion height as the elevation change due to diffusion per unit of t_c

$$h_D = D\nabla^2 z\, t_c = (D\, t_c)\,\nabla^2 z = (K^{-1}D^n U^{1-n})^{2/(n+2m)}\,\nabla^2 z = l_c^2\,\nabla^2 z \,. \tag{A20}$$

Note that the dimensionless diffusion height is:

$$\frac{h_D}{h_c} = \frac{D\nabla^2 z\, t_c}{h_c} = \frac{\nabla^2 z}{U/D} = \frac{\nabla^2 z}{\kappa_c} \,. \tag{A21}$$

10   Multiplying Eq. (A1) by t_c and substituting from Eqs. (A14), (A17), and (A20) we obtain

$$(\partial z/\partial t)t_c = -KA^m(|\nabla z|)^n\, t_c + D\nabla^2 z\, t_c + U\, t_c \iff (\partial z/\partial t)t_c = -h_I + h_D + h_c \,, \tag{A22}$$

which in steady state becomes

$$0 = -h_I + h_D + h_c \,. \tag{A23}$$

At points of zero curvature ($\nabla^2 z = 0$), Eq. (A23) becomes

$$h_I = h_c \iff A^m(|\nabla z|)^n\,(K^{n+2m-2}D^{n-2m}U^{2-2n})^{1/(n+2m)} = (K^{-2}D^{n-2m}U^{2-n+2m})^{1/(n+2m)} \iff$$

$$A^m(|\nabla z|)^n = U/K \,. \tag{A24}$$

Therefore, h_c and U/K are the steady state values of h_I and $A^m(|\nabla z|)^n$, respectively, at points of zero curvature. For $m = 0.5$

15   and $n = 1$, i.e., for the governing equation in the main text (Eq. 1), these two relations coincide.

At drainage divides ($A = 0$), Eq. (A23) becomes

$$h_D = -h_c \iff D\nabla^2 z\, t_c = -U\, t_c \iff \nabla^2 z = -U/D = -\kappa_c \,. \tag{A25}$$

Substituting the definition of h_D (Eq. A20) into Eq. (A23) yields:

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$$h_I = l_c^2 \nabla^2 z + h_c . \tag{A26}$$

i.e., h_I remains a linear function of curvature in steady state. Thus, h_I can be used for valley definition for the generic model in the same way that it can be used for the simplified model.

A5   Length and time scales

The celerity of the incision term is:

$$c(A, |\nabla z|) = K A^m (|\nabla z|)^{n-1} . \tag{A27}$$

5   Therefore, the time needed to advect perturbations over some distance l defines an incision time scale:

$$t_I = \frac{l}{K A^m (|\nabla z|)^{n-1}} . \tag{A28}$$

The diffusion time scale of Eq. (A1) is the same as in the case of Eq. (1), i.e. according to Eq. (35), it is $t_D = l^2/D$.

Therefore, the Péclet number is defined as

$$\text{Pe} = t_D/t_I = \frac{A^m (|\nabla z|)^{n-1} l}{D/K} . \tag{A29}$$

10   We observe that $l_c^{2m+1} G_c^{n-1} = D/K$. Therefore, the Péclet number definition becomes:

$$\text{Pe} = \frac{A^m (|\nabla z|)^{n-1} l}{l_c^{2m+1} G_c^{n-1}} = \left(\frac{A}{A_c} \right)^m \left(\frac{|\nabla z|}{G_c} \right)^{n-1} \left(\frac{l}{l_c} \right) . \tag{A30}$$

The condition $\text{Pe} \approx 1$ is satisfied at points with length scale l, drainage area A, and slope $|\nabla z|$ such that:

$$\text{Pe} \approx 1 \iff \left(\frac{A}{A_c} \right)^m \left(\frac{|\nabla z|}{G_c} \right)^{n-1} \left(\frac{l}{l_c} \right) \approx 1 \iff A^m |\nabla z|^{n-1} l \approx l_c^{2m+1} G_c^{n-1} . \tag{A31}$$

One combination of length scales, drainage areas, and slopes that satisfies this condition is $l \approx l_c$, $A \approx A_c$, and $|\nabla z| \approx G_c$. Substituting these values into the definitions of t_I and t_D yields $t_I \approx t_c$ and $t_D \approx t_c$.

15   ## Appendix B: Rescaling of landscapes

We outline an analytical proof of the rescaling property implied by the dimensionless governing equation (Eq. 16), namely that landscapes with any parameters will evolve temporally and geometrically similarly and will reach geometrically similar steady states provided that we properly rescale their boundary and initial conditions. Equations (18 a–d) defines what we term as proper rescaling.

20

A simple way to demonstrate this property is to explore the necessary conditions for the dimensionless governing equation (Eq. 16) to describe two different landscapes and then show that the same conditions lead to temporally and geometrically similar evolution of these two landscapes. Let a landscape have parameters K, D, and U and a second landscape have parameters $K' = kK$, $D' = dD$, and $U' = uU$, where k, d, and u are positive real numbers, and let both landscapes satisfy

Eq. (16). In what follows, we denote variables and scales of the second landscape as primed to match the notation of its parameters.

First, we derive rescaling relationships between characteristic scales of the two landscapes as functions of k, d, and u. The
5 second landscape's characteristic length scale, l_c', will be

$$l_c' = \sqrt{D'/K'} = \sqrt{dD/kK} = \sqrt{d/k}\ l_c\,. \tag{B1 a}$$

Likewise, its remaining characteristic scales will be

$$h_c' = (u/k)\,h_c\,, \quad t_c' = (1/k)\,t_c\,, \quad A_c' = (d/k)\,A_c\,, \quad G_c' = \left(u/\sqrt{dk}\right)G_c\,, \quad \kappa_c' = (u/d)\,\kappa_c\,. \tag{B1 b–f}$$

Second, we derive relationships between coordinates of the landscapes and between moments in time during their evolution. We substitute Eq. (B1 a) into Eq. (18 a) and obtain

$$x' = (l_c'/l_c)\,x = \sqrt{d/k}\ x\,. \tag{B2 a}$$

10 Likewise, we can obtain

$$y' = \sqrt{d/k}\ y\,, \tag{B2 b}$$

and

$$t' = (1/k)\,t\,. \tag{B2 c}$$

We consider the two landscapes to be temporally and geometrically similar if the points (x, y) and (x', y') have elevations z and z' at moments in time t and t', respectively, such that

$$z'(t') = (u/k)\,z(t)\,. \tag{B2 d}$$

15 Third, one can show that drainage areas, time derivatives, and differential operators of the two landscapes will be related according to

$$A' = (d/k)\,A\,, \quad \partial/\partial t' = k\ \partial/\partial t\,, \quad \nabla' = \sqrt{k/d}\ \nabla\,, \quad \nabla'^2 = (k/d)\,\nabla^2\,. \tag{B2 e–h}$$

Fourth, we can retrieve the following dimensional governing equation for the second landscape if we start from the dimensionless Eq. (16) and use the rescaling formulas of Eq. (18):

$$\frac{\partial z'}{\partial t'} = -K'\sqrt{A'}|\nabla'z'| + D'\nabla'^2 z' + U'\,. \tag{B3}$$

20 We can retrieve Eq. (1), the first landscape's dimensional governing equation, by reversing the derivation of Eq. (16) presented in Sect. 2.2. In Eq. (B3) it is important to note that we use primed elevations, time, drainage areas, and differential operators. This is the necessary condition for the dimensionless Eq. (16) to describe both landscapes, and we found it by using Eqs. (B2 a–h) to rescale variables between the two landscapes.

25 Fifth, we can now show that rescaling according to Eqs. (B2 a–h) additionally leads to temporal and geometric similarity. Specifically, we can show that if the two landscapes are geometrically similar (i.e., they obey Eqs. B2 d) at any moments in time t and t' that are rescaled according to Eq. (B2 c), then they will remain geometrically similar for all later moments in time. To show this we compare the left-hand side and the incision, diffusion, and uplift terms of the right-hand side of the dimensional governing equations of the two landscapes. These terms have dimensions of H T^{-1}, i.e., dimensions of a vertical
30 velocity. Thus, to simplify notation, we denote them here as v, v_I, v_D, and v_U, respectively. The comparisons show that

28

$$v_I{}'(t') = -K'\sqrt{A'}|\nabla'z'(t')| = -kK\sqrt{(d/k)\,A}\,\sqrt{k/d}\,\left|\nabla\big((u/k)\,z(t)\big)\right| = -u\,K\sqrt{A}|\nabla z(t)| = u\,v_I(t)\,, \tag{B4 a}$$

likewise,

$$v_D{}'(t') = u\,v_D(t)\,, \quad v_U{}'(t') = u\,v_U(t)\,, \tag{B4 b–c}$$

and, therefore, also

$$v'(t') = v_I{}'(t') + v_D{}'(t') + v_U{}'(t') = u\left(v_I(t) + v_D(t) + v_U(t)\right) = u\,v(t)\,. \tag{B4 d}$$

Thus, after a time interval dt' the second landscape will have elevation equal to

$$z'(t' + \mathrm{d}t') = z'(t') + v'(t')\mathrm{d}t' = (u/k)z(t) + u\,v(t)\,(1/k)\mathrm{d}t = (u/k)(z(t) + v(t)\mathrm{d}t) = (u/k)z(t + \mathrm{d}t)\,, \tag{B5}$$

i.e., the two landscapes will continue obeying Eq. (B2 d) and thus remain geometrically similar.

5

Therefore, we can conclude that if the two landscapes have geometrically similar initial conditions, then they will evolve temporally and geometrically similarly and will reach geometrically similar steady states.

Appendix C: Numerical simulation set-up

C1 Inputs

10 ### C1.1 Calculation of CHILD parameters

To simulate the governing equation (Eq. 1) with CHILD we used the detachment-limited module, constant, uniform, and continuous precipitation, zero infiltration, hydraulic geometry scaling exponents ω_b and ω_s equal to 0.5, and detachment capacity exponents m_b, n_b, and P_b equal to 1 (see Tucker et al., 2001, and Tucker, 2010, for definitions of CHILD's assumptions, modules, and parameters).

15

For this choice of exponents, the rate of elevation change due to incision is calculated by CHILD from the equations (in CHILD notation)

$$\left.\frac{\partial z}{\partial t}\right|_{\text{Incision}} = -D_c = -k_b\tau\,, \tag{C1 a}$$

$$\tau = k_t\frac{\sqrt{P}\sqrt{A}}{k_w}S\,, \tag{C1 b}$$

where D_c is the maximum detachment capacity in ma^{-1}, τ is stream power per unit bed area in Wm^{-2}, k_b is a detachment rate coefficient in m a^{-1} (W m^{-2})$^{-1}$ (i.e., k_b is the rate of elevation change per each unit of stream power per unit bed area), k_t is the
20 specific weight of water in N m^{-3}, P is the precipitation intensity in m a^{-1}, k_w is bankfull width per unit scaled discharge in s$^{0.5}$ m$^{-0.5}$, and S is slope (Tucker et al., 2001; Tucker, 2010).

Equating the incision term of Eq. (1) to D_c, we can relate the incision coefficient, K, with CHILD's parameters according to

$$K = \frac{k_b k_t\sqrt{P}}{k_w}\frac{\sqrt{1\,\mathrm{a}}}{\sqrt{31557600\,\mathrm{s}}}\,. \tag{C2}$$

CHILD input files use a mixed system of units which includes both years and seconds and the program converts variables to a single system of units. Therefore, to calculate the values of parameters entered into CHILD's input files, we must include the unit conversion factor seen in Eq. (C2).

5   We varied the values of K by varying the value of k_b according to Eq. (C2), while we used constant values of $k_t = 9810 \text{ N m}^{-3}$, $P \approx 1.31 \text{ m a}^{-1}$, and $k_w = 10 \text{ s}^{0.5}\text{m}^{-0.5}$.

We defined combinations of the parameters K, D, and U, each ranging by two orders of magnitude around arbitrary baseline values. The baseline values were found to be used frequently in the literature (e.g., Perron et al., 2008; Tucker, 2009; Clubb
10   et al., 2016). Table 2 shows values of parameters, characteristic scales, and simulation properties of the three landscapes presented in Figs. 1–5 and 7–10, including the baseline landscape (landscape B).

C1.2   Simulation domain and mesh

We synthesized two random TINs by randomly perturbing a deterministic TIN generated by the geometry definition module of MATLAB's PDE toolbox. We used Matlab to better control the rescaling procedure that we describe below. We exported
15   the rescaled TINs of simulations as text files. Using ~~passed~~ the coordinates of TIN points included in these files, ~~to~~ CHILD ~~and allowed it~~ calculated the corresponding Delaunay triangulation using its own, built-in modules.

We set all four domain boundaries to be open (see Tucker, 2010).

20   To prepare rescaled TINs, we followed the following procedure. First, we synthesized two random TINs on rectangular domains with x-coordinates between 0 and 200 and y-coordinates between 0 and 400; the average triangle edge length was 0.8, and the z-coordinates were drawn from a uniform distribution between 0 and 0.1. By assuming that the aforementioned values are unitless, we can consider the resulting TINs to be dimensionless. Second, using Eqs. (3) and (4), we calculated the characteristic length and height scales l_c and h_c that correspond to the parameter combination of each simulation. Third, using
25   Eq. (6), we multiplied the x-, y-, and z-coordinates of the points of the dimensionless TINs by the l_c and h_c of each simulation. This yielded each simulation's dimensional TINs with size of 200 l_c by 400 l_c, an average triangle edge of 0.8 l_c, and initial elevations between 0 and 0.1 h_c. CHILD assumes that x, y, and z are in units of meters; therefore, we calculated l_c and h_c in units of meters as well. The above procedure resulted in dimensional TINs that are rescaled copies of each other according to Eqs. (18 a–c). ~~We used a rectangular domain with side lengths of 200 l_c and 400 l_c, and an average TIN edge~~
30   ~~length $\overline{L_{ij}}$ of 0.8l_c, which led to TINs with approx. 150,000 points.~~

C1.3   Time steps

We defined the time step length Δt using Courant-Friedrichs-Lewy criteria for the incision and diffusion terms of Eq. (1) according to the formulas (Refice et al, 2012):

$$\Delta t = \min\left[\Delta t_{Inc.}, \Delta t_{Diff.}\right],$$ (C3 a)

with

$$\Delta t_{Inc.} = \min\left[\frac{L_{ij}}{K\sqrt{A_i}}\right],$$ (C3 b)

$$\Delta t_{Diff.} = \min\left[\frac{L_{ij}^2}{2D}\right], \tag{C3 c}$$

where L_{ij} is the length of the TIN edge connecting points i and j and A_i is the drainage area of point i. Normalizing lengths and drainage areas in Eqs. (C3 b, c) by l_c and l_c^2, respectively, yields

$$\Delta t_{Inc.} = \min\left[\frac{l_c L_{ij}^*}{K\sqrt{l_c^2 A_i^*}}\right] = \frac{1}{K}\min\left[\frac{L_{ij}^*}{\sqrt{A_i^*}}\right] = t_c \min\left[\frac{L_{ij}^*}{\sqrt{A_i^*}}\right] = t_c \Delta t_{Inc.}^*, \tag{C4 a}$$

$$\Delta t_{Diff.} = \min\left[\frac{l_c^2 L_{ij}^{*2}}{2D}\right] = \frac{1}{K}\min\left[\frac{L_{ij}^{*2}}{2}\right] = t_c \min\left[\frac{L_{ij}^{*2}}{2}\right] = t_c \Delta t_{Diff.}^*. \tag{C4 b}$$

Therefore, both the incision and diffusion time steps limits are in effect rescaled by t_c, even though we do not explicitly impose such scaling.

5

CHILD allows the definition of a single time step length, which is used throughout the entire simulation. However, since A_i evolves during simulations we do not know a priori which combination of L_{ij} and A_i will yield the smallest $\Delta t_{Inc.}$. Therefore, to err on the safe side, we calculate $\Delta t_{Inc.}$ using the domain-wide minimum of L_{ij} and an A_i equal to $0.5(100l_c)^2$, which is the area of a square whose diagonal spans the domain's half width. We assume that this square has a similar area as the largest basins to be formed. To avoid very small L_{ij} which would result in very short Δt, we do not allow TIN perturbation during the synthesis of random TINs (described above) to result in L_{ij} shorter than 1/3 of the average TIN length of $0.8l_c$.

C1.4 Steady state

We assume that $\varepsilon = 0.001$ in Eq. (19), i.e., we assume that a steady state has been reached when

$$\max\left[\frac{|\Delta z_i|}{\Delta t}\right] \le 0.001U, \tag{C5}$$

where Δz_i is the elevation change of point i during a time step Δt.

15 ## C2 Outputs

CHILD produces output files with various variables. Relevant to our model are those with data of elevation, z_i, drainage area, A_i, slope, $|\nabla z_i|$, and stream power per unit stream bed area, τ_i, at every point i, the point IDs of TIN triangle and edge vertices, and the ID of the neighbor to which each point drains.

20 Using the triangle and edge output data, we define the Voronoi polygon associated with a point i, and calculate this point's curvature $\nabla^2 z_i$ and elevation change rate due to diffusion v_{D_i} according to the formulas (Tucker et al, 2001)

$$\nabla^2 z_i = -\frac{1}{\Lambda_i}\sum_{j=1}^{M_i}\frac{z_i - z_j}{\lambda_{ij}}w_{ij}, \tag{C6}$$

$$v_{D_i} = D\nabla^2 z_i, \tag{C7}$$

where Λ_i is the area of the Voronoi polygon of point i, j are neighbors of i, M_i is the number of these neighbors, z_i and z_j are elevations of points i and j, λ_{ij} is the distance between i and j, and w_{ij} is the length of the Voronoi polygon edge shared between i and j.

Author contribution

NT derived analytical results, and NT and JWK interpreted them. NT designed, performed, and analyzed numerical simulations, and NT, HS, and JWK interpreted them. NT drafted the manuscript, and NT, HS, and JWK edited it.

Competing interests

5 The authors declare that they have no conflict of interest.

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Tables

Table 1: List of symbols (in the order they appear)

Symbol	Dimensions (L: length, H: height, T: time)	Description	Definition or first use
(x, y)	L	Horizontal coordinates	Eq. (1)
z	H	Elevation	Eq. (1)
t	T	Time	Eq. (1)
K	T^{-1}	Incision coefficient	Eq. (1)
D	$L^2 T^{-1}$	Diffusion coefficient	Eq. (1)
U	$H T^{-1}$	Uplift rate	Eq. (1)
A	L^2	Drainage area	Eq. (1)
$|\nabla z|$	$H L^{-1}$	Topographic slope	Eq. (1)
$\nabla^2 z$	$H L^{-2}$	Curvature	Eq. (1)
m	[-]	Drainage area exponent	Eq. (A1)
n	[-]	Slop exponent	Eq. (A1)
ε	$H T^{-1}$	Steady-state threshold	Eq. (2)

l_c	L	Characteristic length	Eq. (3)
h_c	H	Characteristic height	Eq. (4)
t_c	T	Characteristic time	Eq. (5)
$(x^*, y^*), z^*, \nabla^*$, etc.	[-]	Dimensionless variables, operators, etc.	Eq. (16)
i, j	L	Unit vectors	Eq. (9)
G_c	$H\,L^{-1}$	Characteristic gradient	Eq. (11)
κ_c	$H\,L^{-2}$	Characteristic curvature	Eq. (13)
A_c	L^2	Characteristic area	Eq. (15)
K', x', l_c', etc.		Parameters, variables, scales, etc., of the second of a pair rescaled landscapes	Eq. (18)
u_c	$L\,T^{-1}$	Characteristic horizontal velocity	Eq. (20)
h_I	H	Incision height	Eq. (21)
h_D	H	Diffusion height	Eq. (22)
k_s	H	Steepness index	$k_s = A^{m/n}|\nabla z|$
κ_{thr}	$H\,L^{-2}$	Curvature threshold for valley definition	Sect. 4.1.4
h_{thr}	H	Incision height threshold for valley definition	$h_{thr} = l_c^2\,\kappa_{thr} + h_c$
Pe	[-]	Péclet number	Eq. (36)
c	$L\,T^{-1}$	Kinematic wave celerity	$c = K\sqrt{A}$
l	L	Length scale	Sect. 4.2
t_I	T	Incision time	Eq. (34)
t_D	T	Diffusion time	Eq. (35)
p	[-]	Exponent of drainage area in scaling relationship with flow path length	Sect. 4.2.3

Table 2: Parameters and resulting characteristic scales of the three landscapes presented in Figs. 1–5. Landscape B, the baseline landscape, has parameters with typical values (e.g., Perron et al., 2008; Tucker, 2009; Clubb et al., 2016). We use the baseline landscape to demonstrate properties of height and length scales in Figs. 7–10.

Parameters, Characteristic scales	Units	Landscape A	Landscape B (Baseline)	Landscape C
K Incision coefficient	a^{-1}	10^{-5}	10^{-6}	10^{-7}
D Diffusion coefficient	$m^2\,a^{-1}$	10^{-3}	10^{-2}	10^{-1}
U Uplift rate	$m\,a^{-1}$	10^{-5}	10^{-4}	2.5×10^{-5}
$l_c = \sqrt{D/K}$ Characteristic length	m	10	100	1,000
$h_c = U/K$ Characteristic height	m	1	100	250

$t_c = 1/K$ Characteristic time	a	10^5	10^6	10^7
$A_c = l_c^2$ Characteristic area	m^2	10^2	10^4	10^6
$G_c = h_c/l_c$ Characteristic gradient	m m^{-1}	0.1	1	0.25
$\kappa_c = S_c/l_c = h_c^2/l_c$ Characteristic curvature	m m^{-2}	10^{-2}	10^{-2}	2.5×10^{-4}

Figures

Figure 1: Horizontal similarity of rescaled landscapes. Steady-state shaded-relief maps demonstrate the horizontal geometric similarity of three landscapes with widely varying parameters but properly rescaled domains (see Eq. (18) for definition of proper rescaling). We label axes in units of each landscape's characteristic length l_c (on the bottom and right in normal fonts) and in kilometers (on the top and left in bold fonts) to highlight the similarity of the three landscapes despite their very different sizes. The landscapes are shown in order of increasing l_c from left to right. Notice that the sizes of the three simulated domains differ by factors of 10 in units of kilometers, but are identical in units of l_c. Lengths and heights scale separately. This leads to different characteristic gradients G_c across landscapes, manifested as varying grayscale intensity ranges. Despite these pronounced differences in gradients, the three landscapes are geometrically similar in planview.

Figure 2: Vertical similarity of rescaled landscapes. Steady-state elevation maps and transects demonstrate the vertical geometric similarity of the three landscapes of Fig. 1. We color the maps by elevation with color scales that are rescaled by each landscape's characteristic height h_c using a colormap distributed with the SIGNUM model (Refice et al., 2012). Notice that across color scales each color corresponds to the same elevation value in units of h_c, but to different elevation values in meters. Therefore, the fact that the color patterns are identical reveals the vertical similarity of the landscapes. Transects corroborate the vertical similarity. Transects pass through the highest peak of the landscapes and are marked on the maps with thick black lines. Note that elevations measured in units of h_c (in normal fonts) are the same, while elevations measured in meters (in bold fonts) are different. The landscapes are shown in order of increasing l_c from left to right. (The ranges of elevation in map color scales and transect z axes match those of Figs. 4 and 5.)

Figure 3: Temporal similarity of evolving, rescaled landscapes. We compare the evolution of the three landscapes of Fig. 1 using shaded-relief maps drawn at four properly rescaled moments in time (see Eq. (18 d) for definition of proper rescaling of time). The comparison shows that, at rescaled moments in time, the horizontal patterns of the landscapes are geometrically similar (and geometrically identical in units of l_c). Each row shows four snapshots of a given landscape. The fourth column shows steady-state landscapes, i.e., those of Fig. 1. The snapshots that appear in each vertical column correspond to the same moment in rescaled time. Values of time in units of t_c are the same along each column (labels in normal fonts), while the values of time in years vary (labels in bold fonts). Time increases from left to right, and horizontal scale increases from top to bottom. Lengths and heights scale separately; this leads to different characteristic gradients G_c across landscapes, manifested as varying grayscale intensity ranges across rows.

Figure 4: Temporal, horizontal, and vertical similarity of evolving, rescaled landscapes. We compare elevation maps of the three evolving landscapes of Fig. 2, using the same snapshots as in Fig. 3 and the same layout (i.e., landscapes sorted by row, rescaled times sorted by column). Color maps are rescaled by each landscape's characteristic height, h_c. The comparison illustrates the horizontal and vertical geometric similarity of the landscapes at rescaled moments in time. For each landscape (i.e., across each row) we use a single color scale, constant in time, to show how elevations (rescaled by each landscape's characteristic height h_c) evolve. The fact that the color patterns are identical within each column reveals the vertical component of the temporal and geometric similarity of the landscapes. Thick black lines mark the transects shown in Fig. 5. The fourth column shows steady-state landscapes, i.e., those of Fig. 2.

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Figure 5: Temporal and vertical similarity of evolving, rescaled landscapes. Transects of the three evolving landscapes of Fig. 2 corroborate the vertical component of their temporal and geometric similarity. We use the same snapshots as in Figs. 3 and 4 with the same layout (i.e., landscapes sorted by row, rescaled times sorted by column). The rescaled transects are identical along each column, demonstrating the exact temporal and geometric similarity of the rescaled landscapes. Transects pass through the highest peak of the steady-state landscapes and are marked on the maps of Fig. 4 with thick black lines. The fourth column shows steady-state transects, i.e., those of Fig. 2.

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Figure 6: Schematic illustration of height scales. Profiling the incision, diffusion, and characteristic height scales h_I, h_D, and h_c along a flow path allows visualizing their properties as described by Eqs. (27)–(29). Part (a) shows the height scales of incision, diffusion, and uplift (h_I, h_D, and h_c, corresponding to the solid, dashed, and dotted black lines, respectively) along a flow path from the drainage divide to a valley. Part (b) shows these height scales as changes in elevation along a steady-state elevation profile (thick green line). Subtracting h_I from the elevations along the profile, or adding h_D or h_c to them, shows the change in elevation per unit of characteristic time t_c that would result from incision, diffusion, and uplift (the solid, dashed, and dotted gray lines, respectively). At the divide (point P_1), incision is ineffective and diffusion balances uplift ($h_I = 0$ and $h_D = -h_c$; Eq. 28). At the point where curvature is zero (point P_2), net diffusion is zero and incision balances uplift ($h_D = 0$ and $h_I = h_c$; Eq. 27). (Note that P_2, where $\nabla^2 z = 0$, generally is not the same as the inflection point of the profile line.) Along the entire profile, the combination of incision and diffusion balances uplift; thus, the distance between the h_I and h_D lines in part (a) (solid and dashed black lines) is constant and equal to h_c ($h_I - h_D = h_c$; Eq. 29).

Figure 7: Steady-state relationship between drainage areas, slopes, and curvatures, parameterized by characteristic length and height scales l_c and h_c. Plotting incision height data ($h_I = \sqrt{A}|\nabla z|$) versus curvature data ($\nabla^2 z$) from the simulated steady-state landscape B (shown in Figs. 1–5), shows that these data followed a linear trend consistent with the relation $\sqrt{A}|\nabla z| = l_c^2 \, \nabla^2 z + h_c$ (Eq. 31). We label incision height and curvature axes in units of m and m m^{-2}, respectively, in bold fonts, and in units of characteristic height h_c and characteristic curvature κ_c, respectively, in normal fonts. Vertical and horizontal dashed lines facilitate the calculation of the slope and intercept of this trend. For every unit of κ_c of curvature increase, the product $\sqrt{A}|\nabla z|$ increases by a unit of h_c; thus, the trend's slope is $h_c/\kappa_c = l_c^2$. To the curvature value $\nabla^2 z = 0$ corresponds the value $\sqrt{A}|\nabla z| = h_c$; thus, the trend's intercept is h_c. The linear relationship illustrated here shows that we can view the characteristic length and height l_c and h_c as parameters of a steady-state relationship that constrains the topographic properties A, $|\nabla z|$, and $\nabla^2 z$. This relationship must be satisfied so that incision, diffusion, and uplift are in equilibrium. Thus, we can view h_c as a scale that expresses the competition between these three processes. (Note the two arrows that mark where data from points P1 and P2 of Fig. 6 would plot.)

Figure 8: Steady-state valley networks visualized by incision height h_I. The simulated equilibrium landscape B (shown in Figs. 1–5) is shown here with each pixel colored by the incision height $h_I = \sqrt{A}|\nabla z|$. In steady state, this incision height is linearly related to topographic curvature (Eq. (33), Fig. 7), with values of $h_I < h_c$ corresponding to convex topography (ridgelines) and values of $h_I > h_c$ corresponding to concave topography (valleys). Thus, high values of h_I reveal the dendritic valley network.

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Figure 9: Prediction of ridge migration by differences of incision height h_I across ridges. These maps show that migrating ridges move toward the side with relatively smaller h_I values. The top row shows four snapshots during the early phase of the evolution of landscape B (also shown in Figs. 1–5). The red squares seen in the top row are focused on a ridge that migrates from left to right; they are also shown magnified in the bottom row. The dashed black lines inside the red squares are fixed at the initial position of the migrating ridge to more clearly illustrate its movement. The black arrows in the bottom row point to the direction toward which the ridge will migrate. In all maps, each pixel is colored by the incision height $h_I = \sqrt{A}|\nabla z|$. Darker colors correspond to higher values of h_I. These maps show that drainage basins expanded at the expense of neighboring drainage basins with lighter colors, i.e., with relatively lower h_I values. For the case of Eq. (1), the incision height h_I is equal with the steepness index k_s, which was among the metrics used by Whipple et al. (2017) to quantify erosion rates. Therefore, these maps confirm Whipple et al.'s finding that the direction of ridge migration can be predicted by erosion rate differences across migrating ridges.

Figure 10: Steady-state valley networks visualized by Péclet number Pe. The simulated equilibrium landscape B (shown in Figs. 1–5) is shown here with each pixel colored by the logarithm of $Pe = \sqrt{A}\, l / l_c^2$ (Eq. 36), where l is the flow path length (the maximum distance along flow paths from each pixel to the drainage divide). The Péclet number quantifies the relative horizontal influence of incision versus diffusion. Thus, dark pixels in this map are horizontally dominated by incision (i.e., valleys) and light pixels by diffusion (i.e., ridgelines).

Figure 11: Dependence of drainage area on flow path length and convergence or divergence of topography. Three points that have equal flow path lengths can have very different drainage areas. Elevation contours (gray lines) reveal that point P_1 is in a valley with convergent topography, P_2 on a hillslope with planar topography, and P_3 on an interfluve with divergent topography. The thick dashed line shows the ridge line and the three dotted lines show flow paths from the ridge to the points P_1, P_2, and P_3. The three flow paths have equal lengths l. The three brown polygons show the contributing areas that correspond to a given contour width (thick black bars) centered at the three points. Point P_1 has a much larger drainage area A than points P_2 and P_3. Note the topography of the contributing area of point P_1; it tends to become planar near the ridge. Specifically, the lateral boundaries of the contributing area tend to become parallel, and the curvature of the contour lines decreases. This example highlights that, in general, contributing areas have mixed topographies.

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