Dear Reviewers,

thanks for your constructive and encouraging comments, in particular to the two anonymous reviewers who made many suggestions to improve the accessibility of the manuscript to a wider readership. It is good to see that the problem with the grid-resolution dependence is still an important issue. In the following, the points addressed in your reports are discussed, and changes to the manuscript are described. Line numbers refer to the version with highlighted changes.

Reviewer 1

"A point of clarification: Paragraph 23 of Perron et al. (2008) reads, ... So the analysis in that paper uses an approach similar to that of Howard (1994) and does consider the physical meaning of the channel width, even if the numerical experiments assume it is spatially uniform!"

I think I got the subpixel approach of Howard's model and the meaning of the channel width w correctly. However, my key point is that your version with constant channel width is in principle correct, but somehow for the wrong reason as the length scale that is needed for compensating the mesh width δ is not the channel width, but another property related to the threshold catchment size where fluvial erosion starts, and that this property is indeed constant over the drainage network. **I explained this point more clearly in the revised version by extending the description of the problem (Sect. 2) and the discussion (Sect. 5)**.

Reviewer 2

"Before providing my comments on the manuscript, I wish to first review a key alternative approach to the problem as a means of introducing the general issues at play. Pelletier (2010) addressed the problem of grid-resolution dependence in coupled hillslope-channel landscape evolution models ... It is important to note that such a modification to the colluvial deposition term is not some indirect way of scaling the fluvial term as Hergarten implies. Far from being a "problem obviously coming from the fluvial incision term" (line 60), it addresses a limitation of the model to represent the cross-valley curvature and the effect of that colluvial deposition rates in valley bottoms."

I must admit that my discussion of Pelletier's (2010) approach was way too short. I do not want to raise any doubts against the major part of this paper addressing flow routing and distinguishing between channelized flow and parallel flow. However, I am not convinced by the scaling approach itself, i.e., rescaling the divergence of the flux from the hillslopes. If my understanding and my own analysis are not completely wrong, this approach suffers from the same problem as Howard/Perron version. As soon as river width increases with catchment size, river steepness is no longer consistent with the empirical findings of Hack (1957) unless the exponent m is changed. In this case, however, the relationship to the widely used concept of the erodibility is lost. According to my findings, this problem affects both approaches suggested previously in almost the same way. The numerical example with the parallel rivers given by Pelletier (2010) navigated around this problem by considering relief and valley spacing. **I discussed the problem more thoroughly in Sect. 2 now (lines 86–114).**

"Grid-resolution dependence in coupled colluvial-fluvial models can be seen most readily as a dependence of drainage density on pixel size."

I do not fully agree to this statement. If we assume that rivers start at points with a given minimum catchment size A_c (in m^2, not in DEM pixels) and a well-organized dendritic network (not parallel flow on slopes), the dependence of drainage density on DEM resolution is rather weak. It is rather the total area covered by the DEM pixels. This should be clearer now in the **more detailed explanation (lines 153–171).**

"If I understand correctly, Hergarten is proposing to use this variation/error in drainage density to scale the fluvial erosion term."

The dominator is indeed something like drainage density except for two differences: (i) Area is not total area as it is in drainage density, but only the part of the area not draining to leaves of the river network. This is the part that makes my analysis a bit complicated at first sight. (ii) Total river length is area of the DEM that covers the network divided by mesh width. For a square grid, this means that diagonal river segments have the same length as those in direction of the axes. This should also be clearer with the **new explanation (lines 153–171).**

"I am wary of this approach because there is no clear (at least to me) physical basis for why the fluvial erosion term would need to be scaled in this way and because there is no indication that the drainage density predicted by the model, even if it can be shown to be grid-resolution independent, is the correct one for a given set of model parameters after such scaling."

I would immediately buy this argument if the widely used model was derived from physical principles. Then the rivers would not know about properties such as drainage density. However, it comes from empirical data of "typical" rivers eroding "typical" landscapes. My conjecture is that the expression for the fluvial erosion law, in particular the value of the erodibility, refers to an equilibrium of erosion and uplift in the catchment and does not describe the river as an isolated object.

"I apologize if I missed it, but I didn't see that Hergarten demonstrated that his approach actually leads to grid-resolution-independent results. I was expecting to see model results with similar topography as the pixel size varies over a wide range. No such figure appears in the paper. I recommend that Hergarten present such a figure along with any other analysis (e.g., predicted steady state drainage density as a function of pixel size) needed to demonstrate grid-resolution independence of the model predictions. I would like to see such grid-resolution independence also demonstrated for cases on non-uniform uplift rates, as such applications are common in landscape evolution models."

I am afraid that you did not miss it. I only thought about the theoretical concept and the generic hillslope process model that follows the direction of the river pattern where it is somehow clear that it should work. You are right, it is not so clear, in particular if we use a "realistic" hillslope process model. **I have now added a new section (Sect. 4) with 2 numerical examples of different complexity.**

"I had a hard time following the description of the scaling approach. My understanding is that the hillslopes and channels in the model output are first differentiated using a user-defined threshold area, A_c, and then the fluvial erosion term is modified by an amount equal to a power-law function of A_c. The power-law modification to A_c is clear but how is A_c chosen? Does the model have to be run first without scaling the fluvial erosion term in order to determine A_c and then rerun with the scaling?"

Sorry for this! It is much easier than you think, and the practical relevance of the value of A_c is limited in most applications. The result of my approach is that the erodibility K as it is usually considered is not the parameter that we need, but K multiplied by a length constant (which is not the river width) instead. I suggest $\sqrt{2A_c}$ as a simple estimate of this length scale. If we use a given erodibility K, we expect a certain channel steepness in equilibrium with a given uplift rate. The only prediction of my concept is that we can define any value A_c and let fluvial erosion act only at catchment sizes $A > A_c$, we will arrive at the correct channel steepness. In many applications there will be hillslope processes affecting scales larger than A_c. If these are strong, fluvial erosion will lose relevance even for for $A > A_c$, and the value of A_c also becomes less relevant. If it is much smaller than the scale of the hillslope process, it even only defines the reference topography that would occur if the considered hillslope process was switched off. I hope that this has become clearer in the revised version, in particular with the help of the **numerical example in Sect 4.**

"Please provide a step-by-step guide for performing the proposed scaling that is applicable not just to the case of steady uniform uplift to steady state but for other potential landscape evolution model applications. It may be that for the case of steady uniform uplift, channels and hillslopes can be differentiated based on a threshold contributing area, but many landscape evolution models are of non-uniform uplift and hence non-uniform drainage density. Moreover, there is a large literature on how to differentiate hillslopes and channels both in models and real-world DEMs, and the use of a single contributing area threshold is universally regarded as an inadequate approach to such differentiation. Assuming that choosing A_c involves differentiating hillslopes and channels before scaling the fluvial erosion term, this manuscript glosses over a very complex topic, the implications of which likely influences the applicability of the proposed method."

Not really – it is all only about bringing empirically determined values of K into the model. We are free to assume any model for fluvial erosion at small scales such as a spatially variable threshold or a continuous decrease of erosion rates at decreasing catchment sizes. We just have to keep in mind that the value of K is the one that we would measure from equilibrium river profiles if we assume that fluvial erosion is switched on for $A > A_c$. In this approach, differentiating hillslopes from channels on a given topography would only be useful if we want to use a specific value of K measured in a given catchment. If we knew the spatial distribution of erosion in this catchment, we could use it for assigning a "realistic" value of A_c to this value of K. However, this is hopeless in most cases, so that we have to accept the problem that measured values of K many unresolved dependencies (including something like A_c) as you already mentioned.

"A minor issue: it is incorrect to state that the erodibility coefficient K depends on rock characteristics and precipitation (line 25). K is influenced by any factor other than channel slope and contributing area that influences detachment-limited erosion rates, including channel width, all of the factors that influence rainfall-runoff partitioning (including vegetation, soil texture, the distribution and sequence of storm events), snowmelt dynamics (for some catchments), etc."

Finally, at least one point where I agree without any reservations. **I have streamlined the wording (lines 27–29).**

Reviewer 3

"(1) The scaling problem: the paper describes in words the scaling problem as it manifests in landscape evolution models, but a picture would be worth a thousand words. I think the impact of the paper would be greater if the author added a figure showing visually the effect discussed section 2: the steepening of topography with decreasing pixel size. Pelletier (2010) has a figure showing the (relative) lack of such effects when using his proposed solution, so one idea would be to mirror that figure (his Fig 6) but without any attempt to scale the problem away. You could even use the same parameters. I would also suggest including a plot showing equilibrium slope-area scaling for models at different pixel resolutions."

I must admit that my explanation of the problem was way too short and written for readers who are already familiar with the problem. **I have now extended Sect. 2 and introduced 3 new figures. The new Figure 3 gives a very simple example of the problem. This scenario is used in the following to explain why both approaches suggested before do not solve the problem completely and is also the basis for the first numerical example presented in the new Sect. 4.**

"Then, follow up by showing the same examples, but now using the proposed scaling solution. This would (presumably) demonstrate that the solution works. Adding such a 'before and after' pair of figures seems really key to selling the core idea of the paper; otherwise, readers might be left wondering 'if I bother to do this, will it really work?'."

I have now added a new section (Sect. 4) with two numerical examples. The first one continues the simple example from Sect. 2 and shows that the approach works perfectly here not only for the steady-state solution, but also concerning the time scale. The second example combines diffusion with fluvial incision and can be seen as some kind of standard scenario for coupling fluvial and hillslope processes. Here the main result is that the approach reduces the scaling problem considerably, but drainage reorganization by hillslope processes leaves a part of the scaling problem.

"(2) Some parts of the paper come across as if written for readers who are already well familiar with the relevant literature. I recommend making a few small additions / modifications that would make the paper accessible to, for example, graduate students who are just starting out, or people in other fields who have a new interest in the topic. Mostly this is a matter of adding example references to the literature and/or expanding on some points, as noted in the specific comments below.'

I tried to make it accessible to a wider readership now – for details see below.

"(3) A general question, which would be worth answering somewhere in the text, is whether the scaling analysis still holds if the drainage patterns are qualitatively different on hillslopes versus channels. In this Discussion version of the paper, it is not clear whether the simulation in Figure 1 includes any local transport (ie hillslope) processes; if so, it is not apparent in the drainage patterns."

The entire framework was indeed developed for the situation that the fluvial drainage pattern persist on the hillslope as if fluvial erosion was the only erosion process. This also applies to Fig. 1. **I tried to point this out more clearly in the revised version and hope the second numerical example in Sect. 4 shows where the remaining problems are.**

"19-20 For readers unfamiliar with this idea, it would be helpful to add one or a few example references (one of the earliest I am aware of is Andrews and Bucknam, 1987; another option would be to cite a review paper that discusses mathematical representations of various geomorphic processes)"

Andrews and Bucknam (1987) is also the earliest reference to this that I know, although its relationship to coupled models is not really close. **I have added this one together with those that I often use as key references in this context (line 21).**

"23 'has become some kind of paradigm' - this will not mean much to readers who are just joining the conversation. Suggest giving one or a few example references."

Here I would not fully agree. I am quite sure that even most of the readers just joining the conversation about combining fluvial and hillslope processes have either read a at least one paper about modeling fluvial erosion or at least one where the erodibility as a lumped parameter is discussed. In both cases, the chance is quite high that these readers have already seen Eq. (2) or be able to find a paper where it is explained in detail.

"24 I disagree that equation 2 (I think that is what is meant by 'it') requires the assumption of constant precipitation. ... In any event, the text about 'constant precipitation' (constant in space or time or both?) seems like just a side comment, and maybe the best approach would be simply to delete it."

Indeed only a side comment, although I would not expect any reader to run into problems with this statement. **I have removed it (lines 24–25).**

"26 As above, I would argue it is the spatial distribution of runoff that matters most; precipitation has some influence on this, but there are other factors too."

This is, of course, all true. Originally I just wanted to point out that the erodibility is not only a property of the rock as the term could suggest, but a lumped parameter. **I streamlined this point (lines 27–29).**

"33 The relation predicts m/n = theta ONLY if the erosion rate and erodibility are uniform in space and steady in time. You allude to that in the next sentence, but the way this is worded would be confusing for a reader who does not understand that you are referring to a special case here. I recommend re-wording this section to be more precise."

The sentence should not imply that rivers follow Eq. (3) locally then, but only that the empirical finding of Eq. (3) in many rivers constrains the ratio $\frac{m}{n}$. **I clarified it in order to avoid any confusion (lines 36–37).**

"35-38 I would argue that the condition of equilibrium is more general than the word 'uplift' implies. The key is that the erosion rate is space-time uniform. This could be due to actual tectonic uplift relative to, say, sea level. Or it could be an equilibrium relative to a given rate of base-level lowering at the boundary of a given system (and in fact the former is a subset of the latter)."

Of course, but if we replace U with E, Eq. (4) is not even restricted to spatially uniform conditions, but just Eqs. (2) and (3) combined. **However, I have replaced U with E in Eq. (4) now.**

"42-43 'the total area covered by large rivers decreases with decreasing mesh width': can you provide evidence for this, or otherwise clarify this concept? ... Maybe what you actually mean here is that the surface area covered by stream segments ('channel pixels'), rather than drainage area, shrinks as pixel size shrinks (tending toward zero when the network segments become infinitesimally wide linear features)."

Yes, of course! Maybe the problem is that I worked with concepts such as box counting in the context of fractals too long, so that I could not imagine that anyone could misunderstand this point. I hope the **rephrased version is clearer (lines 46–47).**

"46 - I think there is a bit more to it than that. If you omit local transport (ie, diffusion or diffusion-like modification of the topography), you have the odd circumstance where for the equilibrium case the equations predict that $H \to \infty$ as $A \to 0$. In practical terms, then, a model with just eq 2 would have increasing relief with decreasing pixel size. Might be worth pointing out, as the current text ('scaling problem may not be critical') could be misinterpreted as meaning there is no pixel size dependence without local transport."

The revised version describes the scaling properties of the version without local transport in more detail, including two new figures. I would, however, prefer to get around the question whether $H \to \infty$ for $A \to 0$. $S \to \infty$ is clear, but if we use Hack's (1957) scaling relation between A and upstream length, H remains finite.

"58-61 I don't think this summary quite does justice to Pelletier (2010). I suggest adding something like 'where the factor is unity on cells identified as hillslopes, but greater than unity for cells that represent valley features'. Also, you might add that a reason to suspect it doesn't work for 'all types of local transport' is that his derivation was (heuristically at least) based on a linear model."

Yes, the discussion of the approaches by Howard/Perron et al. and Pelletier was indeed much too short. **I have now added a more thorough discussion of the properties of both ideas and, in particular, why Pelletier's approach does not solve the problem completely (Sect. 2).**

"62 Can you expand here to say why large mesh widths would be immune? Large relative to what? Is the idea that if all cells are conceptually valley cells, then you don't need special treatment for hillslopes versus valleys?"

My impression is that it is practically like this. In some studies (including some own), the mesh width is large enough to assume that all sites are channel sites, and hillslope processes are just a small add-on to make the topography more realistic. If we then do not compare simulations with different resolutions and do not mind whether the channel steepness is as expected, it is tempting to disregard the problem. **The revised version addresses this aspect more precisely (lines 115–124).**

"75 For what it's worth, I would argue that 'bedrock incision' just means what it says, and does not (or at least should not) imply any particular mechanism or model thereof. I think the idea you are trying to get at here is that there is a difference between assuming that a channel must entrain and remove only the material on the channel bed, or that it must entrain and remove that plus the sum of material transported into the channel from surrounding hillslopes. I do not think the term 'bedrock erosion' is all that helpful in articulating the difference between these two possibilities, but it would be worth expanding on the idea: for example to note that it depends on the degree of contrast between the 'mobile' material coming from side slopes and the 'intact' or 'original' material in the channel floor (one example of highly resistant material coming from side slopes is Shobe et al. (2016 GRL))."

As far as I can see, the term 'bedrock erosion' did not occur; it would indeed be confusing. I must admit that I did not get the point 'bedrock incision just means what it says'. For me it somehow implies that bedrock at the location of the river is eroded, and this is not a particular mechanism or model for me. I still think that the terms 'bedrock incision' and 'detachment-limited erosion' reflect the differences between the two concepts quite well. Nevertheless I agree that **including the reference Shobe et al. (2016) is a good idea (lines 133–135).**

"86-88 Consider noting here that Pelletier (2010) described an alternative approach based on comparing computing drainage area on the DEM grid, and on a 2x higher resolution interpolated version of the DEM. That approach has the advantage of allowing the processes to determine the drainage density. I suspect that the mechanism for identifying channel versus hillslope pixels probably does not matter much for the technique you propose, and if that is the case, then it would be worth pointing out. For the sake of developing the idea, using a fixed A_c seems totally fine. But as a reader would like to know whether I can still use the approach if I use a different method for distinguishing channel and hillslope pixels."

This is indeed a central point and probably the main limitation of the concept. I hope that this limitation **becomes clearer in the discussion section (Sect. 5) now.**

"89-90 I got confused at first by the definition of A_e. A key aspect of the definition is that it includes only those pixels that drain DIRECTLY to a given channel pixel, and not ones that 'pass through' another channel pixel upstream. If that understanding is correct, it would be worth stating this (because other readers, like me, are probably used to thinking of contributing area as something that accumulates downstream). Eq 5: unless you are changing the definition of K, this equation seems to change the meaning of E: in eq 2 it seems to be length per time, but in eq 5 it seems to become volume per time. If that is correct, I recommend using a different symbol than E to avoid confusion. Note I am assuming that A_e is a surface area. The text says 'number of sites', so I guess it is actually meant to be dimensionless (just a count), as text later in the paper implies. But in that case then you're no longer talking about a physical law. Why not treat A_e as a surface area, and either have the equation represent the volumetric erosion rate over the area concerned, or divide by cell area to arrive at a length per time. At any rate, clarification of these issues in the text would be helpful."

Ok, I remember that you said that a picture would be worth a thousand words and hope that my **more detailed explanation helps in combination with the new Fig. 4. I also hope that it is clearer now that all areas are measured in DEM pixels throughout Sect. 3.**

"94 - It would be very helpful to add more information about this model and the conditions under which it was run to generate figure 1. Is OpenLEM in this example solving just eq 2 or does it include diffusion too? What flow routing algorithm does it use? How are closed depressions handled? Was it run until steady state balance between erosion and uplift/baselevel was reached? Does the fluvial threshold A_c actually apply in the numerical model, ie, are areas smaller than A_c treated exclusively with local transport? Is local transport applied to all pixels or just those $A < A_c$? Or, alternatively, was the model run without any threshold or hillslopes? In addition, please list all the input parameters so readers could reproduce or replicate the experiment."

Ok, I have now **included this information, but quite at the beginning of the paper as the new Fig. 1 already uses topographies of the same type.** Except for filling local depressions. These are considered as lakes (deepest outlet) for computing the flow pattern, and erosion is switched off as long as the water level is higher than to topography. However, this is relevant only in the very beginning of the simulations as these lakes vanish soon. I would therefore prefer not to confuse the readers with this additional information.

"121 I think you mean 'site' not 'size' "

Thanks!

"120-125 and eq 6: I found this section confusing. I understand A_e to be a spatial field, with a different value at every pixel. Yet if $P(A_c)$ is just a scalar fraction, then eq 6 implies a unitary value for A_e. Is your aim here a derived distribution of the cumulative probability of A_e? ... Ah ok, reading later, you mention A_e is dimensionless (but perhaps you can see why it is confusing given that A and A_c refer to areas)."

I hope that **this is clearer now.** A_e defined as the mean size of the contributing areas (in pixels) over all channel sites with a given catchment size. Then the main point is that it is almost independent of the considered catchment size and only depends on A_c. The confusion with the areas measured in pixels should hopefully dissolve now, too.

"134 I recommend a more extensive explanation here. Clearly figure 4 shows that the A_e-A_c relation follows a power law with about the same slope as that of the cumulative area distribution. But the underlying scaling argument is hard to follow."

I would say it is an immediate consequence of Eq. 12 (numbering of revised manuscript). the argument why it is not exactly the same may indeed be more complicated, but I think it is not a problem for the following parts of the manuscript if it is not immediately understood.

"151 It would be helpful to know the parameters used to generate these synthetic topographies."

I thought the readers would guess that everything remains the same except for those parameter values explicitly mentioned in Table 1. But in order to state it clearly, **I added a sentence (line 237).**

"eq 9: if you used this directly in a model, would it not break equilibrium slope-area scaling? Or are you suggesting that the leading factors compensate for deposition by material sourced from surrounding hillslopes? ... This is basically the argument you're making, right? That effectively a fluvial grid cell has to drill through not only its own material, but also all the material coming from the surrounding hills. I think the idea would be conveyed more clearly if you added some math along the lines of the above."

Yes, exactly like this! I hope that this point becomes clearer with the more detailed explanations in the previous parts, so that there should be no need to extend it here.

"183-4 reference for this number?"

A good chance to promote my own paper **(line 277).**

"185-eq 10: I can see the advantage of this approach, but would like to see some discussion of how to reconcile the concept of a threshold area A_c with the actually valley head area that emerges from a model. To mirror my questions above, are you suggesting that this approach should be paired with using a model that only applies fluvial erosion to locations with $A > A_c$, where A_c is a parameter? Or could one allow A_c to emerge from the dynamics, as in Pelletier (2010)?"

Yes, indeed with A_c as a parameter. Allowing it to emerge from the dynamics would require the consideration of a given hillslope process in detail. I hope this becomes clearer with the **second numerical example (lines 300–341).**

"205-6 good point, and some models I'm aware of allow for diffusion-like transport to be applied ONLY to convex locations, with the assumption that the material is instantly carried away in concave-up locations."

In some sense "Make it as simple as possible, but not simpler." But what is possible for mountain streams?

"Code availability: I do not know what the policy of Esurf is, but 'available on request' is no longer generally considered best practice. Better to place code in a community repository, or at least a public repository. Better still to have it under version control. Even better yet to provide input files, examples of usage, etc., in an open repository (see Wilson et al. below). ..."

I am not sure either, but as far as I know, it is less strict than for AGU journals. To my experience, codes deposited in repositories are not as valuable as it seems unless someone keeps maintaining it continuously. In case of OpenLEM I am not even able to provide enough support for a very limited number of users. **I prepared a repository and placed it for the moment at http://jura.geologie.uni-freiburg.de/esurf-2019-77.zip**. Looks as if it takes some time to get a permanent repository at our university this time. And I am also afraid that people will be able to reproduce the results, but not much more.

Best regards,
Stefan Hergarten

Rivers as linear elements in landform evolution models

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Abstract. Models of detachment-limited fluvial erosion have a long history in landform evolution modeling in mountain ranges. However, they suffer from a scaling problem when coupled to models of hillslope processes due to the flux of material from the hillslopes into the rivers. This scaling problem causes a strong dependence of the resulting topographies on the spatial resolution of the grid. A few attempts based on the river width have been made in order to avoid the scaling problem, but none of them appears to be completely satisfying. Here a new scaling approach is introduced that is based on the size of the hillslope areas in relation to the river network. An analysis of several simulated drainage networks yields a power-law scaling relation for the fluvial incision term involving the threshold catchment size where fluvial erosion starts and the mesh width. The obtained scaling relation is consistent with the concept of the steepness index and does not rely on any specific properties of the model for the hillslope processes.

1 Introduction

Fluvial incision is a major if not even dominant component of long-term landform evolution in orogens. When modeling fluvial erosion, restriction to the detachment-limited regime considerably simplifies the equations. Here it is assumed that the erosion rate at any point of a river can be predicted from local properties such as discharge and slope, while sediment transport is not considered. The generic differential equation for the topography $H(x_1,x_2,t)$ of a landform evolution model with detachment-limited fluvial erosion reads

$$\frac{\partial H}{\partial t} = U - E - \mathrm{div}\boldsymbol{q} \tag{1}$$

where U is the uplift rate and E the rate of fluvial incision. The third term describes a local transport process at the hillslopes where $\boldsymbol{q}$ is the flux density and div the 2D divergence operator. Linear diffusion is the simplest model here; it was considered in the context of landform evolution by Culling (1960) even before models of fluvial erosion came into play. However, there are also more sophisticated models for $\boldsymbol{q}$ taking into account the nonlinear dependencies of hillslope processes on topography (e.g., Andrews and Bucknam, 1987; Howard, 1994; Roering et al., 1999).

Concerning the fluvial incision term E, assuming a power-law function of the catchment size A and the channel slope S,

$$E = KA^mS^n, \tag{2}$$

has become some kind of paradigm. The parameter K is denoted erodibility. ~~Since this relation should in principle rather involve discharge instead of catchment size, it is only an approximation for constant precipitation, and K is~~ It is a lumped parameter ~~combining rock properties with precipitation. However, writing Eq. (2) in terms of discharge instead of catchment sizeis quite straightforward~~subsuming all influences on erosion other than channel slope and catchment size, so that ~~predicting the spatial distribution of precipitation on a changing topography is the only challenge here~~it is not a property of the rock alone, but also depends on climate in a nontrivial way (e.g., Ferrier et al., 2013; Harel et al., 2016).

Equation (2) is often called stream power approach since it can be interpreted in terms of energy dissipation of the water per channel bed area if an empirical relationship between channel width and catchment size is used (e.g., Whipple and Tucker, 1999). However, the idea behind this approach even dates back to the empirical study of longitudinal channel profiles by Hack (1957). In this study, a power-law relationship between channel slope and drainage area was found, often called Flint's law (Flint, 1974). This relationship is nowadays usually written in the form

$$S = k_{\cancel{s}\mathrm{s}} A^{-\theta} \tag{3}$$

where θ is the concavity index and ~~k_s~~ k_s the steepness index. ~~This relation~~ Assuming that Eq. (3) is the fingerprint of spatially uniform steady-state conditions, it predicts $\frac{m}{n} = \theta$ and allows for a convenient interpretation of the erodibility. If local transport (last term in Eq. 1) is neglected, the steepness index ~~of a steady-state river (equilibrium of uplift and fluvial incision)~~ follows the relation

$$k_{\cancel{s}\mathrm{s}}{}^n = \cancel{\frac{U}{K}}\frac{E}{K}. \tag{4}$$

This relation allows for a simple adjustment of the lumped parameter K in such a way that a given channel steepness is achieved ~~in equilibrium with a given uplift~~ at a given erosion rate.

2 The scaling problem

While widely used and in principle simple, all models of the type described by Eqs. (1) and (2) suffer from a scaling problem. Mathematically, the problem is that catchment sizes are not well-defined in the continuum limit as the catchment of each point degenerates to a line. When considered on a discrete grid, rivers ~~become~~ are represented as linear objects with a width of one pixel. Thus, the total ~~area covered by~~ surface area of the pixels covering the network of the large rivers decreases with decreasing mesh width. ~~As a consequence, the area where uplift can be balanced by fluvial erosion at moderate channel slopes also decreases, so that the overall topography becomes steeper.~~

If local transport is not considered, the scaling problem leads to a canyon-like topography where the width of the valleys decreases with mesh width. ~~However, the rivers still follow Eqs. (3) and (4)~~ This behavior is illustrated in Figs. 1 and 2 where two steady-state topographies with mesh widths of $\delta = 0.01$ (100×100 nodes) and $\delta = 0.002$ (500×500 nodes) are considered. All parameter values are set to unity except for $m = 0.5$, so that ~~the scaling problem may not be crucial. But~~ $\theta = 0.5$. The northern and southern boundaries are held at zero elevation, while the western and eastern boundaries are

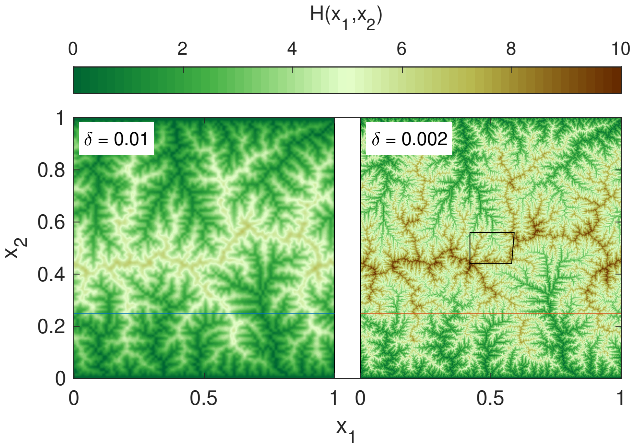

Figure 1. Fluvial equilibrium topographies computed for identical parameter values on grids with different spacing ($\delta = 0.01$, 100×100 nodes and $\delta = 0.002$, 500×500 nodes). The horizontal lines refer to the profiles analyzed in Fig. 2, and the rectangle marks the region shown in Fig. 4.

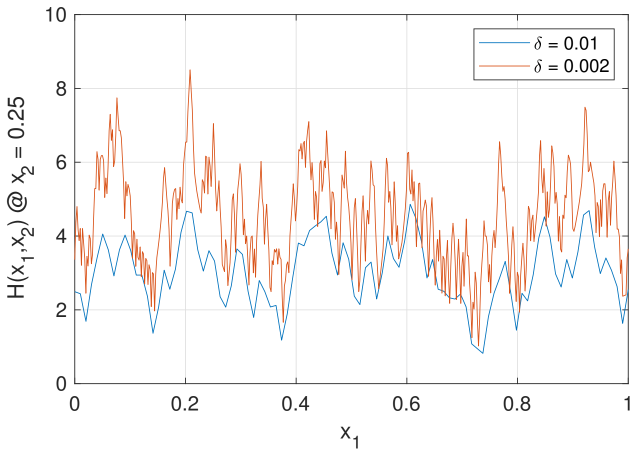

Figure 2. Profiles through the topographies shown in Fig. 1.

periodic. The topographies were obtained from the landform evolution model OpenLEM that was used in some previous studies (e.g., Robl et al., 2017; Wulf et al., 2019), but has not been published explicitly. It uses the D8 flow routing scheme (O'Callaghan and Mark, 1984) and a fully implicit scheme (Hergarten and Neugebauer, 2001; Hergarten, 2002), so that large time steps can be performed in order to ensure that a steady state is achieved. The simulation on the fine grid was started from a flat topography with a small random disturbance, while the simulation on the coarse grid was started from a downsampled version of the finer topography.

Relief increases with decreasing grid spacing because the smallest catchment size that can be resolved is $A_{\text{min}} = \delta^2$, and maximum equilibrium slope is proportional to $A_{\text{min}}^{-\theta} = \delta^{-2\theta}$ according to Eq. (3). As nodes with small catchment sizes can

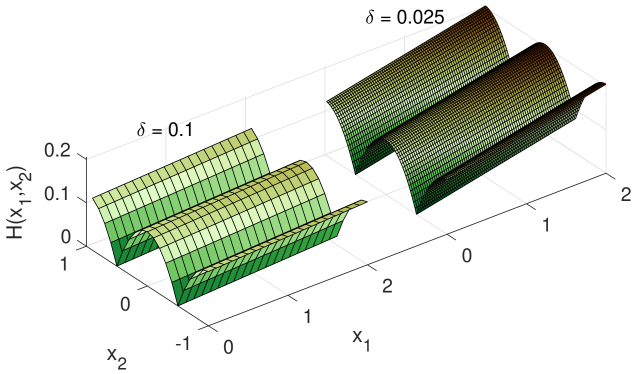

Figure 3. River segments in equilibrium with uplift for different mesh widths δ.

drain directly into large rivers, this increase is not restricted to major drainage divides, but also result in steep valley flanks. The heights of the valley floors are, however, hardly affected by the spatial resolution. Catchment sizes of large rivers even converge in the limit $\delta \to 0$, so that longitudinal profiles of large rivers become stable for $\delta \to 0$ according to Eq. (3). Thus, relief and also mean elevation depend on the spatial resolution for the simplest model without local transport, while large rivers are hardly affected.

The independence of river steepness of resolution is, however, lost as soon as local transport comes into play~~, it also affects the rivers. Then the topography becomes strongly dependent on the mesh width~~. Figure 3 shows the example of short, parallel river segments with unit spacing (periodic in x_2 direction) in equilibrium with constant uplift. Linear diffusion

$$\boldsymbol{q} = -D\nabla H \tag{5}$$

was assumed as the simplest model for local transport. As in the previous example, all parameters except for $m = 0.5$ were set to unity. A catchment size of $A = 10^6$ was assumed for each river segment, so that the channel slope should theoretically be $S = 10^{-3}$ in equilibrium with $U = 1$. While the topography of the hillslopes is in principle independent of the grid spacing δ, the river segment becomes steeper if δ decreases.

~~The problem has been known and addressed~~

The reason for the increasing channel steepness is that the local transport is conservative, so that the river does not only have to incise into the rock at its bed, but also has to remove the material coming from the hillslopes. Regardless of the model used for local transport, a flux of $(d-\delta)U$ per river length enters the site that contains the river in equilibrium where d is the valley spacing. Then the discretized divergence of the flux density is

$$\mathrm{div}\boldsymbol{q} = -\frac{(d-\delta)U}{\delta}. \tag{6}$$

Inserting this result into the steady-state version of Eq. (1) yields

$$E = U - \mathrm{div}\boldsymbol{q} = \frac{d}{\delta}U, \tag{7}$$

so that the fluvial erosion rate required for compensating uplift is by a factor $\frac{d}{\delta}$ higher than it would be without local transport. This requires an increase in channel slope by a factor of $\left(\frac{d}{\delta}\right)^{\frac{1}{n}}$ according to Eq. (2).

This scaling issue has been known for more than 25 years, and two approaches have been suggested to overcome the problem were proposed. Howard (1994) suggested a subpixel representation of the rivers where a river segment only covers a fraction of a grid cell. It was assumed that this fraction is $\frac{w}{\delta}$ where w is the river width~~and δ the mesh width of the grid~~, and then the fluvial incision term E was multiplied with this factor. Perron et al. (2008) transferred this concept to the detachment-limited case. According to Eq. (7), rescaling E by the factor $\frac{w}{\delta}$ yields

$$E = \frac{d}{w}U, \tag{8}$$

so that the dependency on δ indeed vanishes.

While straightforward at first sight, this scaling approach is not free of problems. The channel width in general increases in downstream direction, so that equilibrium river profiles are no longer consistent with Eq. (3)~~if E is rescaled without further modifications.~~. Perron et al. (2008) avoided this problem by assuming a constant channel width and postponing it to subsequent studies. As discussed by Pelletier (2010), taking into account an increase of channel width in downstream direction would require a reduction of the exponent m ~~must be lowered in~~ in Eq. (2) in order to keep it consistent ~~. In this case, the physical unit~~ with Eq. (3). However, unit and meaning of the erodibility K ~~changes, which destroys its relation to the channel steepness. Assuming a constant channel width w (e.g., Perron et al., 2008) introduces a dimensional parameter without a physical meaning and is practically equivalent to replacing K by wK. So the unit of K also changes in principle, and the problem remains basically the same.~~ would change then.

In order to overcome this problem, Pelletier (2010) suggested to leave the fluvial incision term as is and rescale the local transport term ~~(last term in Eq. 1)~~ div$\boldsymbol{q}$ by the inverse factor $\frac{\delta}{w}$ ~~. This formally avoids the problems discussed above, but it might be questionable whether a problem obviously coming from the fluvial incision term can be fixed by rescaling another term in the equation, in particular whether this works for all types of local transport.~~at sites containing rivers. Practically, this rescaling means that the flux of material coming from the hillslopes is not distributed over the entire grid cell, but only over the part of the area covered by the river. So it can be seen as the inverse of the subpixel approach of Howard (1994) and Perron et al. (2008) applied to the local transport instead of the fluvial erosion. For the steady-state example considered above, this rescaling leads to

$$\mathrm{div}\boldsymbol{q} = -\frac{(d-\delta)U}{w} \tag{9}$$

instead of Eq. (6), so that

$$E = U - \mathrm{div}\boldsymbol{q} = \frac{(d+w-\delta)}{w}U. \tag{10}$$

For $w \ll d$ and $\delta \ll d$, however, this relation approaches Eq. (8), so that this concept suffers from the same problem as the approach of Howard (1994) and Perron et al. (2008).

So there seems to be no completely satisfying solution of the scaling problem so far. ~~For large mesh widths δ and for rather qualitative studies (e.g., Wulf et al., 2019), it may not be crucial . However, even the comprehensive study on the scaling behavior~~ Several contemporary modeling studies (e.g., Duvall and Tucker, 2015; Gray et al., 2018; Wulf et al., 2019; Reitman et al., 2019) neither of the two approaches, but implement Eq. (1) as is without taking its dependence on the grid scale into account. This is not a crucial problem as long as simulations with different spatial resolutions are not compared and as long as we are aware that the erodibility K has a limited meaning. As soon as the relevance of fluvial erosion ~~in combination with hillslope diffusion by Theodoratos et al. (2018) disregards the problem by claiming that it dissolves if the entire equation is transformed to nondimensional coordinates. This is , however, not true as the grid spacing δ persists as an additional length scale.~~and hillslope processes is assessed quantitatively or scaling relations are developed (e.g., Theodoratos et al., 2018), the problem may become crucial. A further discussion is given in Sect. 5.

Other recent approaches navigate around the scaling problem by neglecting the flux of material from the hillslopes into the rivers. The recently presented landform evolution model TTLEM (Campforts et al., 2017) makes a distinction by catchment size in such a way that fluvial erosion only acts on sites with a catchment size above a given threshold A_c, while hillslope processes only act at smaller catchment sizes. It is assumed that all hillslope material entering the rivers is immediately excavated without any further effect, so that fluxes from hillslopes into rivers can be disregarded, and the scaling problem does not occur. This approach reduces the interaction between rivers and hillslopes to a one-way coupling where only the rivers have an influence on the evolution of the hillslopes and can be seen as an implementation of bedrock incision in the strict sense. While it seems that the terms detachment-limited erosion and bedrock incision are sometimes used synonymously, it should be clarified that the applicability of the concept of ~~bedrock incision in this strict sense~~ pure bedrock incision is probably much narrower than that of detachment-limited erosion, in particular if highly resistant material is brought into the channels (Shobe et al., 2016). The same in principle holds for the model most widely used in the context of drainage divide migration (Goren et al., 2014) where analytical solutions for hillslope processes are used on the sub-pixel scale.

3 A new scaling approach

The ~~scaling issue can be unraveled by reconsidering the empirical basis of~~ simple example considered in the previous section involves a dependence on grid spacing δ according to the factor $\frac{d}{\delta}$ without rescaling (Eq. ~~(2). If fluxes of material from the hillslopes into the rivers are not disregarded, the equilibrium of uplift and erosion must be reinterpreted. Erosion at the hillslopes must keep up with fluvial incision in order not to form deeper and deeper canyons. Fluvial incision is the only process in~~ 7). Both approaches for rescaling replace the dependence on δ by a dependence on the channel width w, so that a factor $\frac{d}{w}$ remains (Eq. ~~(1)that immediately removes material, while the last term describing local transport is conservative and thus preserves the total volume. Therefore, the volume per time carried away by a river segment is not the product of the rate defined by~~ 8). This is, however, still a problem if w is not constant. The occurrence of the factor $\frac{d}{w}$ suggests that the river spacing d would be a more suitable characteristic length scale for rescaling than w if we want to preserve the form of the erosion law (Eq. 2~~with the area of the river bed, but the product with this area plus a certain hillslope area. An estimate of this area will be developed in~~

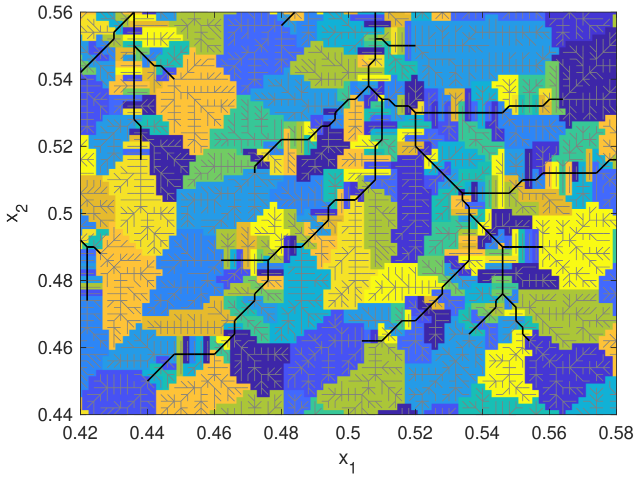

Figure 4. Flow pattern of the central region of Fig. 1. Black lines show rivers with $A \geq A_c$ for $A_c = 100$ pixels. Gray lines are channels with $A < A_c$ considered as hillslope sites. Each colored area consists of one river site plus the hillslope area that drains to this river site.

~~this section, and it will finally yield a new scaling relation for the rate of incision as a function of the grid spacing~~) without changing the exponents m and n. In the following, a concept that generalizes the simple example of parallel rivers to dendritic networks is developed.

~~In a first step, grid cells belonging to river segments must be distinguished from grid cells interpreted as hillslopes where only local transport takes place. The~~

Let us start from the simplest approach to ~~implement such a distinction is~~ distinguish channel sites from hillslopes by defining a threshold catchment size A_c in such a way that all sites with $A \geq A_c$ are river segments, while all ~~grid cells~~ sites with $A < A_c$ ~~are hillslope sites. Then the question is how many hillslopessites deliver their~~ belong to hillslopes. As local transport is conservative, all material eroded anywhere has to be removed by the river sites, so that we need to know how much material each river sites receives from the hillslopes. The area of the respective hillslopes can be determined for a given topography without any specific assumptions on the transport process except for the direction of transport. The simplest model is to assume that local transport follows the hypothetic channel network at the hillslopes, i.e., the direction of steepest descent on a purely fluvial topography. Figure 4 illustrates this concept. Each colored area consists of one channel site and the hillslope area that delivers its eroded material to ~~a given river site. This number plus one (for the river site itself) is the~~ this site.

If the size of this area was the same for each river site, rescaling the fluvial erosion rate (Eq. 2) according to

$$E = A_e K A^m S^n \tag{11}$$

where A_e is the size of this area measured in DEM pixels (i.e., the number of sites~~that have to be eroded by the considered river site. Let us call this number~~) would already solve the scaling problem. However, it is immediately recognized in Fig. 4 that the sizes of these areas are highly variable. A random variation in these sizes is not a problem. If A_e ~~. The expression for the fluvial erosion rate (~~in Eq. (11) is the mean size, channel steepness will just vary randomly, which is also found in nature.

A systematic dependence of the area on catchment size would, however, be a problem. In this case, equilibrium river profiles would be no longer consistent with Eq. ~~2)must then to be modified according to~~

$$E = A_e K A^m S^n.$$

(3), so that the problem would be basically the same as in the previous approach for a non-constant channel width.

In the following, numerically obtained equilibrium drainage networks are analyzed in order to find out how A_e depends on A and on A_c. ~~These networks were obtained from the landform evolution model OpenLEM that was used in some previous studies (e.g., Robl et al., 2017; Wulf et al., 2019), but has not been published explicitly.~~More precisely, A_e is the mean size of all hillslopes areas draining to channel sites with a given catchment size A at a given fluvial threshold A_c (plus the respective channel site). For simplicity, all areas are measured in DEM pixels in the following considerations, i.e., as a number of sites. Starting point of the analysis is ~~a~~ the drainage network of a fluvial equilibrium topography on a square $L \times L$ grid with $L = 10000$~~where the northern and southern boundaries are held at zero elevation, while the western and eastern boundaries are periodic. The simulation was started from a flat topography with a small random disturbance and a constant uplift rate. A concavity index of $\theta = 0.5$ as originally suggested by Hack (1957) and an exponent $n = 1$ were assumed. The drainage network of the obtained steady-state topography is~~. Boundary conditions and parameter values except for the grid size are the same as in the smaller examples shown in Fig. 1.

~~Drainage pattern of a fluvial equilibrium topography computed on a $10,000 \times 10,000$ grid. For clarity, the image was reduced to 2000×2000 pixels taking the highest catchment size within each 5×5 tile.~~

~~Figure 2~~ Figure 5 reveals that the eroded area A_e increases with the fluvial threshold A_c, but becomes independent of A if the catchment size A is sufficiently large. This means that the hillslopes draining to large rivers are not systematically larger than those draining to small rivers. It is the reason why we will arrive at a scaling relation that preserves the form of Eq. (2) and avoids the problem occurring if the river width is used for scaling.

The increase of A_e if A approaches A_c can be explained by distinguishing between river segments and channel heads. Let us define channel heads as those sites without any tributary with $A \geq A_c$, i.e., as those sites that are only supplied by hillslopes. All other sites with $A \geq A_c$ are considered as river segments. All sites with $A = A_c$ are channel heads and thus follow the relation $A_e = A$, so that all curves start at the dotted line in Fig. ~~2~~5. The resulting values A_e of the river segments (without the channel heads) are shown by the dashed lines in Fig. ~~2~~5. The increase of A_e if A approaches A_c even turns into a decrease then. This decrease arises from the limitation $A_e \leq A - A_c$ that holds for all river segments as those have at least one tributary cell contributing at least A_c. So the contribution of the hillslopes must be small if A is only slightly larger than A_c. However, the decrease is exaggerated by the logarithmic scale and concerns only a small number of sites. So it makes sense to assume that A_e is independent of A for river segments.

Both the number of river segment sites and the number of channel head sites decrease with increasing threshold A_c. The decrease of the latter is faster, so that the ratio of the numbers of head sites vs. river sites converges to zero for large A_c. This is, however, not true for the total contributions. Figure ~~3~~ 6 shows the ratio of the sum of the A_e values of all river segments and the sum of the A_e values of the channel heads. It can also be interpreted as the ratio of the total area that must be eroded by the

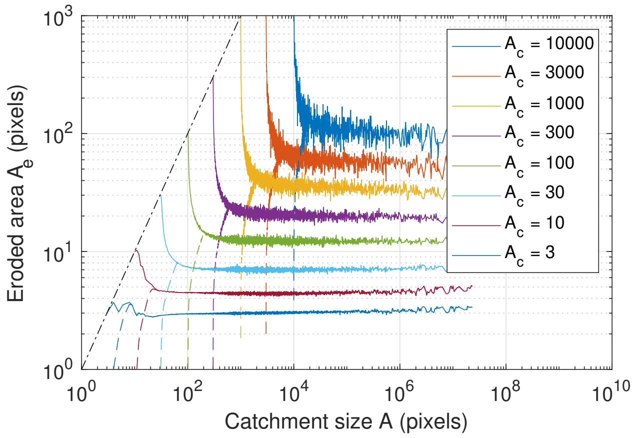

Figure 5. Eroded area A_e as a function of the catchment size A for different fluvial thresholds A_c. Raw data were used for those catchment sizes that occurred at least 1000 times on the grid. Otherwise, data were binned dynamically so that there are at least 1000 points in each bin.

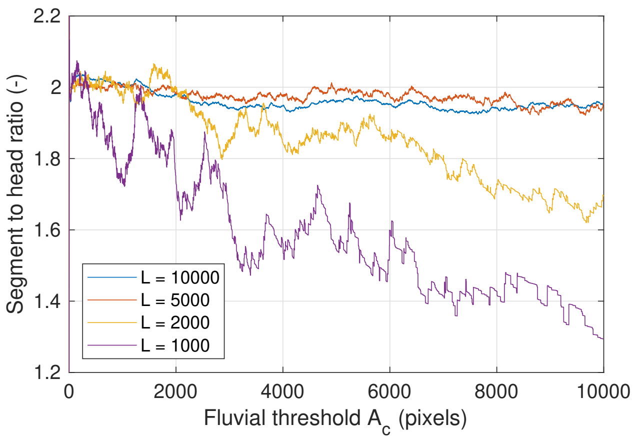

Figure 6. Ratio of total area eroded by all river segments and total area eroded by all channel head sites as a function of the fluvial threshold A_c.

river segments over the total area that must be eroded by the channel heads. The results shown for different grid sizes shown in Fig. ~~3~~ 6 suggests that this ratio becomes constant in the limit of large grid sizes. It apparently approaches a value of about 2 here, which means that the river segments contribute about two thirds to total fluvial erosion and the channel heads one third.

This result suggests that the dependency of A_e on the threshold A_c is determined by the cumulative distribution $P(A)$ of the catchment sizes in the drainage network. This distribution describes the probability that a randomly selected ~~size~~ site has a catchment size $\geq A$. The probability $P(A_c)$ evaluated at the fluvial threshold is the ratio of the area covered by all channel pixels and the total area. It can be interpreted as a drainage density (river length per total area) on a discrete grid. Then a fraction $P(A_c)$ of the considered domain must erode a given fraction (here about two thirds) of the ~~total considered~~ domain,

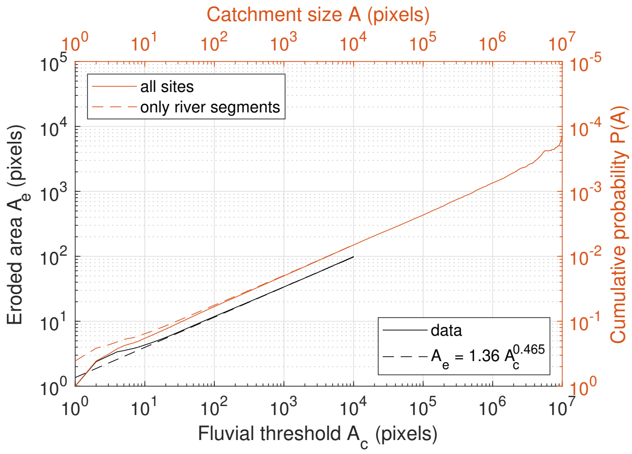

Figure 7. Black axes: eroded area as a function of the fluvial threshold. Red axes: cumulative distribution of the catchment sizes.

leading to the relation

$$A_\mathrm{e} = \frac{\gamma}{P(A_\mathrm{c})} \tag{12}$$

with $\gamma \approx \frac{2}{3}$ for this network. While A_e can be measured directly for the considered drainage network, its relation to $P(A)$ (Eq. 12) is useful as this distribution has already been investigated in several studies on natural and modeled drainage networks (Rodriguez-Iturbe et al., 1992a; Maritan et al., 1996b; Rodriguez-Iturbe and Rinaldo, 1997; Rinaldo et al., 1998; Hergarten and Neugebauer, 2001; Hergarten, 2002; Hergarten et al., 2014, 2016). It was found that $P(A)$ follows a power-law distribution

$$P(A) \sim A^{-\beta} \tag{13}$$

over a reasonable range where a range $\beta \in [0.41, 0.46]$ was found except for the two latest studies. In these studies, larger networks were considered making use of increasing data availability and computing capacities. An exponent very close to 0.5 was found for both optimal channel networks (OCNs, see below) (Hergarten et al., 2014) and a real river pattern at the continental scale (Hergarten et al., 2016).

Equations (12) and (13) suggest a power-law relation

$$A_\mathrm{e} = \alpha A_\mathrm{c}^{\beta} \tag{14}$$

between the eroded area and the fluvial threshold. The validity of Eqs. (12), (13), and (14) is ~~validated~~ investigated in Fig. ~~4~~7. Comparing the two solid curves reveals that Eq. (12) does not hold exactly since the curves come closer to each other for decreasing catchment sizes. The reason is that A_e only refers to the river segments without the channel heads, so that $P(A_\mathrm{c})$ in Eq. (12) should also exclude the channel head sites. The dashed red line in Fig. ~~4~~ 7 showing the accordingly reduced distribution $P(A)$ illustrates that Eq. (12) indeed holds then, and that the effect vanishes for large A_c.

The black dashed line in Fig. ~~4~~ 7 refers to the best-fit power-law relation according to Eq. (14). It is based on all integer values of A_c from 1 to 10,000 assuming equal errors, so that the large values of A_c practically have a high weight in the fit.

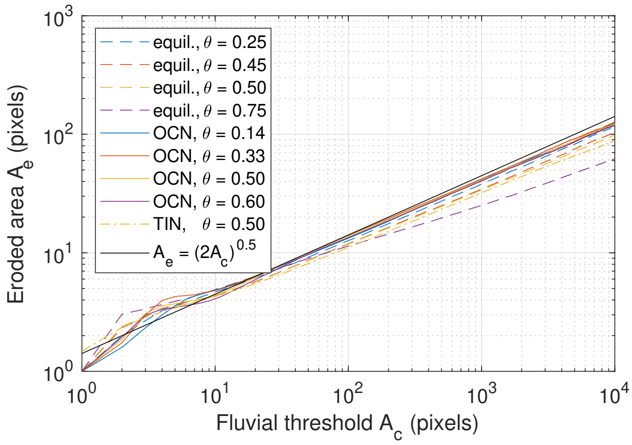

Figure 8. Eroded area A_e as a function of the fluvial threshold A_c for the considered drainage networks. For clarity, only the results obtained from the largest domains are plotted.

The power law with the obtained values $\alpha = 1.360$ and $\beta = 0.465$ fits the data well with a relative error of less than 5 % for $A_c \in [15, 10000]$ and less than 1 % for $A_c \in [400, 10000]$. The deviations are larger for smaller fluvial thresholds due to the fact that dendritic networks cannot be represented well on a regular lattice at small scales.

The relation to the catchment-size distribution (Eqs. 12 and 13) suggests that the power-law dependency of A_e on A_c (Eq. 14) should be universal. For testing this hypothesis, a set of equilibrium topographies with $\theta \in \{0.25, 0.45, 0.5, 0.75\}$ was analyzed. These values cover the range that has been found so far under relatively homogeneous conditions (e.g., Robl et al., 2017). The value $\theta = 0.45$ was added as it is often used as a reference value instead of $\theta = 0.5$ (e.g., Whipple et al., 2013; Lague, 2014). Parameter values and boundary conditions are the same as in the previous example. Since the exponent n has no immediate effect on equilibrium topographies, values $n \neq 1$ were not considered.

The power-law parameters α and β obtained from equilibrium topographies on different lattice sizes L are given in Table 1. In addition, the original data for the largest grids are shown in Fig. ~~5~~8. The results are overall similar with a tendency to lower exponents β for increasing θ. A notable deviation is only found for the very high concavity index $\theta = 0.75$. Here the slopes become very steep at small catchment sizes, resulting in a slower migration of drainage divides during the simulation (Robl et al., 2017). As a result, the topography reaches a steady state quite soon, so that there is finally less reorganization in the drainage network with regard to the initial random pattern. In this sense, the lower exponents found for $\theta = 0.75$ can be seen as some fingerprint of poorly organized river patterns, but are probably not relevant for the rivers that were the empirical basis of the stream power law. These findings confirm that the concavity index θ has a minor effect on the topology of the drainage networks, although it strongly affects the shape of longitudinal river profiles and thus the topography.

In addition, Table 1 and Fig. ~~5~~ 8 also contain results obtained from optimal channel networks (OCNs) on a grid with $L = 4096$. Optimal channel networks are derived from the principle of minimum energy dissipation and have been widely used in the context of river networks (e.g., Howard, 1990; Rodriguez-Iturbe et al., 1992c, b; Rinaldo et al., 1992; Maritan et al.,

Table 1. Parameter values of the power-law relation between eroded area and fluvial threshold (Eq. 14) obtained from different simulated drainage networks on regular lattices with $L \times L$ nodes.

	θ	L	α	β
steady-state topographies	0.25	5000	1.264	0.492
		2000	1.072	0.511
		1000	1.587	0.470
	0.45	5000	1.273	0.478
		2000	1.586	0.451
		1000	1.047	0.499
	0.50	10,000	1.360	0.465
		5000	1.434	0.459
		2000	1.807	0.423
		1000	1.579	0.440
	0.75	10,000	1.653	0.393
		5000	1.715	0.388
		2000	1.433	0.412
		1000	2.179	0.359
OCNs	0.14	4096	1.487	0.480
	0.33		1.626	0.473
	0.50		1.508	0.478
	0.60		1.521	0.475

1996a, b; Rinaldo et al., 1998). The networks considered here are those shown in Fig. 1 of Hergarten et al. (2014) where θ is related to the parameter n used there by $\theta = \frac{n-1}{n+1}$. The values of A_e of OCNs are overall slightly higher than those of the equilibrium topographies, and the variation with θ is lower. As OCNs are organized more strongly than drainage networks of arbitrary equilibrium topographies, the lower variability among OCNs is not surprising.

Table 2 provides additional results obtained from steady-state topographies on triangulated irregular networks (TINs). Numbers of neighbors, distances to neighbors, and areas of pixels are variable here. The latter are defined by the Voronoi diagram. Nondimensional areas (in DEM pixels) are normalized to the mean pixel size given by $\delta^2 = \frac{A_{tot}}{N}$ where A_{tot} is total area and N the number of nodes. The values listed in the Table 2 and the respective curve in Fig. 8 show that the results obtained from TINs are close to those obtained from regular meshes.

These results suggest to define the values $\alpha = 1.508$ and $\beta = 0.478$ obtained from the OCN with $\theta = 0.5$ as reference values. The question is, however, whether such a precision is useful for applications. In particular, $\beta = 0.5$ would be more convenient than lower values. In the considerations made above, ~~A_e and A_c~~ all areas are measured in DEM pixels and are thus

Table 2. Parameter values of the power-law relation between eroded area and fluvial threshold (Eq. 14) obtained from different simulated drainage networks on triangular lattices with N nodes for $\theta = 0.5$.

N	α	β
2×10^7	1.630	0.433
1×10^7	1.611	0.435
5×10^6	1.264	0.466
2×10^6	1.332	0.454
1×10^6	1.400	0.445
5×10^5	1.432	0.450

nondimensional properties. ~~Defining the grid scale δ as the square root of the area of a DEM pixel (which would be the mesh width for a regular square lattice) and considering~~ Considering A_c as a physical (dimensional) area, A_c has to be replaced by $\frac{A_\mathrm{c}}{\delta^2}$ in Eq. ~~14~~(14). Then the fluvial erosion rate (Eq. 11) turns into

$$E = \alpha \left(\frac{A_\mathrm{c}}{\delta^2} \right)^{\beta} K A^m S^n, \tag{15}$$

so that the fluvial incision term scales like $\delta^{-2\beta}$. For $\beta = 0.5$, the fluvial term scales like $\frac{1}{\delta}$. This is not only convenient, but also leads to basically the same scaling relation assumed by Perron et al. (2008). The only difference is that the term $\alpha\sqrt{A_\mathrm{c}}$ occurring here was interpreted as a channel width w and then assumed to be constant for all rivers, so that it lost its physical meaning. So the new formulation of the fluvial incision term also fixes the concern raised by Pelletier (2010) that led to the alternative formulation where the hillslope transport term was rescaled.

In order to estimate α for $\beta = 0.5$, it is helpful to know in which region of Fig. ~~5~~ 8 we are in typical model applications. A breakdown of Flint's law (Eq. 3) was reported at catchment sizes between between about $0.1\ \mathrm{km}^2$ and $5\ \mathrm{km}^2$ (Montgomery and Foufoula-Georgiou, 1993; Stock and Dietrich, 2003; Wobus et al., 2006). However, channel steepness declines at small catchment sizes, so that this breakdown rather implies that other erosion processes come into play than that fluvial erosion is no longer active. In turn, many small springs in mountain regions have ~~subsurface catchment sizes~~ discharges in the order of magnitude of ~~0.01~~0.1 liters per second (e.g., Hergarten et al., 2016), corresponding to catchment sizes $A < 0.01\ \mathrm{km}^2$, but it is not clear whether the erosive action of the resulting small streams follows Flint's law. Reasonable estimates of A_c are probably between these two ranges. Assuming a spatial resolution of about $100\ \mathrm{m}$ or a bit less, A_c will be in the order of magnitude of ~~10~~ a few to 100 DEM pixels. As illustrated by the black line in Fig. ~~5~~8, $\alpha = \sqrt{2}$ provides a reasonable estimate for this range with simple numbers as $\alpha A_\mathrm{c}^{\beta} = \sqrt{2A_\mathrm{c}}$ then. With this estimate, the scaling factor for the fluvial erosion rate is $\frac{\sqrt{2A_\mathrm{c}}}{\delta}$, and the modified stream-power law for fluvial erosion turns into

$$E = \frac{\sqrt{2A_\mathrm{c}}}{\delta} K A^m S^n. \tag{16}$$

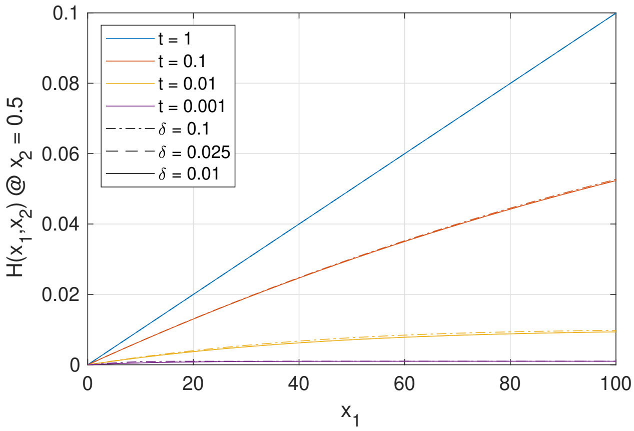

Figure 9. Numerical results for the scenario considered in Fig. 3. The river profiles obtained for $\delta = 0.025$ and $\delta = 0.01$ cannot be distinguished visually.

4 ~~Discussion~~Numerical examples

~~The simple scaling relation for the fluvial erosion rate obtained from simulated drainage networks (Eq~~Let us first return to the example of parallel rivers considered in Fig. 3. It was found in Sect. 2 that the topography of the hillslopes was robust against the spatial resolution, while the channel slope increases with decreasing grid spacing δ. Both approaches previously published fix this problem, but the channel slopes are by a factor of $\frac{d}{w}$ too steep compared to what is expected from the erodibility.

It should be noted that this example is not related to the approach to estimate A_e from A_c for dendritic networks (Eqs. 15 ~~or 16)involves some variations in the parameters due to the topology of the drainage network, but appears to be quite universal. Concerning the hillslope processes,~~ and 16), but can only test the validity of the principal scaling approach (Eq. 11). The size of the area A_e does not follow Eq. (14), but is defined by the geometry as $A_e = \frac{d}{\delta}$ (measured in DEM pixels). Figure 9 shows the numerical results for the parameter values used in Fig. 3 for different values of δ. The simulation was started from a flat topography where the flow paths of the parallel rivers are predefined. As the problem is linear for $n = 1$, this example can also be seen as the change in the river profile through time if uplift suddenly increases at $t = 0$, while the base level remains constant. The results show that the equilibrium profile achieved for large times is reproduced correctly, and that the time-dependent behavior is also robust against the resolution. This means that the scaling approach itself (Eq. ~~(1) only requires that they are conservative. As fluvial equilibrium topographies were used in~~ 11) yields both the correct equilibrium behavior and the correct time scale.

The second example refers to the scenario considered in Fig. 1, but extended by a fluvial threshold $A_c = 10^{-5}$ and by linear diffusion with a diffusivity $D = 10^{-5}$. The threshold A_c is a property of the fluvial erosion process, while the diffusive hillslope process is not related to it. It is thus assumed that fluvial erosion acts only at sites where $A \geq A_c$, while diffusion is active everywhere. A TIN representation is used in order to avoid artefacts from the combination of the ~~simulations, it was, however, implicitly assumed that~~ eight-neighbor (D8) flow routing scheme with the standard four-neighbor diffusion scheme

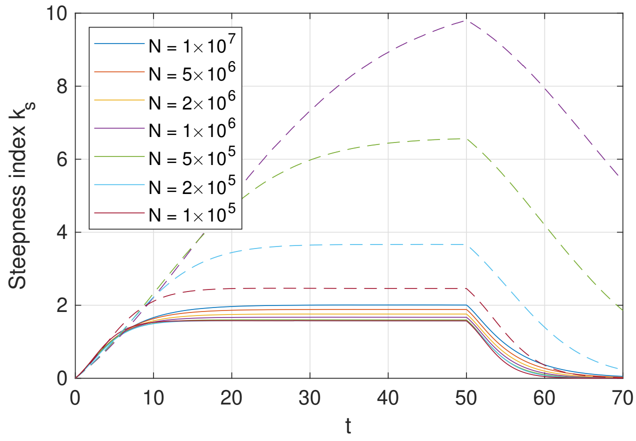

Figure 10. Mean steepness index k_s of the large rivers obtained from simulations on TINs with different resolutions, defined by the total number of nodes N. Solid lines refer to the simplified scaling approach suggested in this paper (Eq. 16), while dashed lines refer to simulations performed without any rescaling. The latter are plotted only for $N \leq 10^6$.

on a regular mesh. The simulations are started from an almost flat topography with unit uplift. Uplift is switched off at $t = 50$ in order to observe the decay of the ~~flux of material on the hillslopes follows the same direction as the flow of water if the hillslopes were part of the fluvial regime.This may not be true since hillslopes are in general smoother than fluvial topographies. However, the scaling relation only depends on the total flux from the hillslopes into the rivers and not on its spatial pattern. Only the relative contributions of river segments and channel head sites may slightly vary, but this should have no big effect. So the scaling relation can indeed be expected to be independent of the specific characteristics of the involved hillslope processes.~~ topography.

~~As the scaling relation originates from the topology of the drainage network, it should also not be limited to the specific form of the stream power law (~~The mean steepness index k_s of the large rivers is plotted as a function of time in Fig. 10. Large rivers are defined by $A \geq 10^{-3}$ here, which is considerably larger than A_c, but much smaller than the domain. As expected, the simulations performed without any rescaling of the erodibility (dashed lines) are strongly affected by the spatial resolution. The steepness index increases with increasing number of nodes N, i.e., with decreasing pixel size. In turn, the results obtained using the simple scaling relation (Eq. 16, solid lines) have a much weaker dependence on resolution. There is, however, a residual variation in channel steepness. The mean value of k_s varies between about 1.6 and 2.0 over the considered range from $N = 10^5$ to $N = 10^7$. This result does not change fundamentally if a higher or lower threshold than $A \geq 10^{-3}$ is used for defining large rivers.

5 Discussion

It may be surprising that the example of fluvial incision and hillslope diffusion considered in the previous section yields a mean steepness index greater than one, although the scaling concept was developed in order to preserve channel steepness.

The concept is, however, based on a generic hillslope process where the direction of transport follows a hypothetic fluvial equilibrium pattern and turns into fluvial erosion at a given threshold catchment size A_c. It is questionable whether any hillslope process occurring in nature comes close to this simple model. In the example considered here, the diffusion process is characterized by a diffusivity D and is not related to A_c. The fluvial domain is affected by diffusion more and more with increasing diffusivity. As a consequence, slopes of small channels decrease, so that they erode less efficiently. This has to be compensated by the larger rivers, so that they must become steeper.

This is, however, a real property of the hillslope process here, and it is not the goal of the scaling approach to remove it. The concept presented here aims at removing the dependence on the resolution and to provide a way how values of the erodibility have to be interpreted. Here it is suggested that they should be considered in combination with a fluvial threshold A_c in such a way that they would yield the expected channel steepness if the generic hillslope model was valid.

In turn, the residual dependence of channel steepness on resolution is a problem, in particular because it is not clear whether it converges in the limit $\delta \to 0$ ($N \to \infty$). The problem arises from network reorganization which also affects the fluvial region. Diffusion disturbs the dendritic topology towards parallel flow where the model based on Hack's findings (Eq. 2) ~~used in the simulations. So it should provide a quite general scaling relation for detachment-limited fluvial erosion.~~ is not valid. Using an improved flow routing scheme that is able to distinguish channelized flow from parallel flow as suggested by Pelletier (2010) and letting A_c self-adjust might reduce the problem. However, the aim of this study is to develop a simple, quite universal rescaling approach that avoids or at least reduces the dependence on resolution without modifying the applied model seriously. In this sense, Eq. (16) should be a good tradeoff.

Nevertheless it is important to keep the difference between detachment-limited erosion and pure bedrock incision in mind. Here it is assumed that the ability of the river to take up particles and carry them away concerns both the river bed and material coming from adjacent hillslopes. If we, conversely, assume that all material coming from the hillslopes is instantaneously removed by the river without any consequences, there is no feedback of the hillslopes to the rivers, and Eq. (1) does not require any rescaling.

The results of this study have consequences for scaling relations in coupled models of rivers and hillslopes. Theodoratos et al. (2018) conc a comprehensive analysis of the problem with linear diffusion without rescaling. The parameters they used were the same as in the previous example (Fig. 10), so that it is immediately clear that their numerical results strongly depend on resolution. The authors argued that, following the approach of Pelletier (2010), both grid spacing and channel width are rescaled, so that the ratio $\frac{\delta}{w}$ remains constant, and the scaling issue is consistent throughout all scales. However, the results presented here show that the property relevant for compensating δ is not channel width, but A_e and thus A_c. These parameters are, however, physical properties of the erosion process, so that they do not scale with the size of the domain. As a consequence, the characteristic horizontal length scale of the coupled system should rather be

$$l_c = \frac{D}{\sqrt{A_c}K} \tag{17}$$

for $m = 0.5$ and $n = 1$ instead of $l_c = \sqrt{\frac{D}{K}}$ used by Theodoratos et al. (2018). This problem also affects the recent extension by an erosion threshold (Theodoratos and Kirchner, 2020).

6 Conclusions

This study presents a simple scaling relation for the fluvial incision term in landform evolution models involving detachment-limited fluvial erosion and hillslope processes. In order to avoid a dependence of the simulated topographies on the spatial resolution of the grid, the fluvial incision term must be multiplied by a scaling factor depending on the ratio of the threshold catchment size A_c where fluvial erosion starts and the pixel size δ^2 of the grid. The analysis of several simulated drainage networks yields a power-law dependence of the scaling factor in Eq. (15) with an exponent slightly lower than 0.5. However, for application in numerical models, a simpler approximation where the fluvial erosion rate is rescaled by a factor $\frac{\sqrt{2A_c}}{\delta}$ is suggested. As this relation assumes a simple, generic hillslope process, it cannot provide an exact solution for all types of hillslope processes. In combination with such processes, e.g., diffusion, the dependence on the spatial resolution is not completely removed. Nevertheless, the simple scaling relation appears to be a reasonable tradeoff between accuracy and simplicity.

Code and data availability. All codes and data can be downloaded from the FreiDok data repository at .... The author will be happy to support interested readers in reproducing the results and performing subsequent research.

Author contributions. N/A

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