Stochastic threshold incision

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Motivation

Modern work on fluvial incision thresholds started with Tucker and Bras [2000], and since then all the major work on fluvial erosion thresholds has included a stochastic treatment [Wolman and Miller, 1960, Tucker and Bras, 2000, Snyder et al., 2003, Tucker, 2004, Lague et al., 2005, DiBiase and Whipple, 2011, Lague, 2014, Rossi et al., 2016, Scherler et al., 2017, Deal et al., 2018]. As Lague [2014] points out, if you try to use a threshold with constant forcing, you have major nonlinearities in steady-state slope, critical discharge, critical area, etc, when the magnitude of the forcing is close the the magnitude of the threshold. There are likely few rivers in the world whose discharge is anything steady enough to be considered remotely constant, and at the same time, it is likely that many rivers have a threshold of motion that is not exceeded at least part of the time. Truly, the concepts of stochastic forcing and fluvial erosion threshold different aspects of the same process.

Applying a stochastic forcing

Start with

$$\frac{\partial z}{\partial t} = U - D\Delta z + E \tag{1}$$

where E is the fluvial incision. To properly do the incision with a threshold, it is critical to take into account the variability in flow. This allows for incision to occur "below" the threshold, so there is not a singularity when $K\sqrt{A}|\nabla z| = \theta$. We can do this by considering a short timescale incision model

$$I = K\sqrt{A}|\nabla z|q^* - K\theta \tag{2}$$

where q^* is the discharge normalized by the mean. The simplest option would be to consider that q^* is exponentially distributed with a mean of 1,

$$p(q^*) = e^{-q^*} \tag{3}$$

Then the fluvial incision rate is

$$E = \int_{q_c^*}^{\infty} I(q^*)p(q^*)dq^* = K\sqrt{A}|\nabla z| \int_{q_c^*}^{\infty} (q^* - q_c^*)\, e^{-q^*} dq^* \tag{4}$$

where q_c^* is found by solving $I(0)$, leading to $q_c^* = \frac{\theta}{\sqrt{A}|\nabla z|}$. The solution to the integral is simply

$$E = K\sqrt{A}|\nabla z| e^{-\frac{\theta}{\sqrt{A}|\nabla z|}} \tag{5}$$

Therefore, the full PDE is

$$\frac{\partial z}{\partial t} = U - D\Delta z + K\sqrt{A}|\nabla z| e^{-\frac{\theta}{\sqrt{A}|\nabla z|}} \tag{6}$$

If we nondimensionalize in the same way as in the paper, this leads to

$$\frac{\partial z^*}{\partial t^*} = 1 - \Delta^* z^* + k_s^* e^{-N_\theta/k_s^*} \tag{7}$$

where $k_s^* = \sqrt{A^*}|\nabla^* z^*|$ and all other parameters are exactly as in the paper. Note, now the PDE is no longer piecewise, but instead there is an exponential decrease to zero in the magnitude of the fluvial incision term in the neighbourhood of $k_s^* \approx N_\theta$.

Possible interpretations of N_θ

You mention that there are 3 characteristic scales in the problem. I think that θ implies a fourth scale, which is fundamentally the characteristic grain size in the landscape. The characteristic grain size D_c sets the threshold of motion. In the simplest case, it will be linearly proportional to the threshold itself [e.g. Scherler et al., 2017], so we can say $D_c \propto \theta$. Many papers on threshold stochastic formulations point out that the ratio of $\frac{\Psi}{U} = \frac{K\theta}{U}$ is the key nondimensional number that describes the importance of thresholds [e.g. DiBiase and Whipple, 2011, Lague, 2014].

I also point out in Deal et al. [2018] that this ratio is similar, and when everything is linear, identical to the ratio of the discharge that can move the mean grain size to the mean discharge $\frac{q_c}{\mu}$. Since the discharge and area are the same here, the mean discharge (cast as a flow depth) is $\sqrt{A}$, and the critical discharge is the flow depth that can move the mean grain size for a given slope, here we take the characteristic slope $\theta = q_c G_c$, so the critical discharge is $q_c = \frac{\theta}{G_c}$. The same result can be found by setting $I(q^*) = 0$ and solving for the critical area at the mean discharge $q^* = 1$ and the characteristic slope G_c. Then we see that if we look at the nondimensional critical discharge at the characteristic area and slope

$$\frac{q_c}{\mu_c} = \frac{q_c}{\sqrt{A_c}} = \frac{q_c}{l_c} = \frac{\theta}{l_c G_c} = \frac{K\theta}{U} = N_\theta. \tag{8}$$

So N_θ can be thought of as referring directly to the channel hydraulics at the characteristic area. In fact there are many perspectives on it: $N_\theta \propto \frac{D_c}{h_c}$ can be thought of as a nondimensional grain size, scaled by h_c; or as a nondimensional critical discharge/flow depth $N_\theta = \frac{q_c}{l_c}$, scaled by l_c; or as a nondimensional critical area to reach the critical discharge at mean flow $N_\theta = \frac{A_\theta}{A_c}$, $(A_\theta = q_c^2)$; or as a nondimensional shear stress $N_\theta = \frac{\theta}{\theta_c}$, scaled by the characteristic shear stress $\theta_c = \sqrt{A_c} G_c$.

One of the main conclusions of earlier studies is that whether grains are considered large or small is a function of the uplift rate. When uplift rates are high, grains are relatively smaller. This is captured very nicely by N_θ.

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