Dear Reviewers, dear Editor,

thanks for your constructive and encouraging comments! In the following, the points addressed in your reports are discussed, and changes to the manuscript are described. Line numbers refer to the version with highlighted changes at the end of this document.

Reviewer 1 (Xiaoping Yuan)

"Assumption in the transport-limit erosion: The author has an assumption to obtain the sediment flux in equations (10) and (11), which are important for the later derivation of implicit and O(n) scheme of the transport-limit erosion model. Erosion rate KA^mS^n in equation (6) is the rate at the outlet of drainage area A. The equation (11) assumes that the sediment flux of the drainage area equals to $KA^mS^n \times A = KA^{m+1}S^n$, which implies that the erosion rate is same everywhere in the drainage area A, which is true at the steady state subjected to a uniform uplift rate. Because at steady state, erosion everywhere balances rock uplift rate such that under conditions of spatially uniform uplift the total sediment flux at a given point along a river equals the product of upstream drainage area (A) and the rock uplift rate or the uniform erosion rate. This has been proved by the author using the uniform uplift rate (Figure 1, left panel) that transport-limited model produces the same, final steady-state landscape as the detachment-limited model. However, the author needs to show several transient-state comparisons between these two models before reaching steady state, and may test the sediment flux out of the domain to explore the differences between these two models (e.g., Armitage et al., 2018, ESurf). I have the feeling that they are different even they have the same final landscape."

Yes, of course! Both end members have not much in common concerning their transient behavior. It is a second-order diffusion equation vs. a first-order advection equation. Transport-limited erosion in principle even supports no distinct transient knickpoints. I thought the difference in transient behavior was clear, and I just wanted to point out with the numerical example that they also differ under non-uniform steady-state conditions. The key point of Sect. 2 is just that the old findings of Hack (1957) could be alternatively be interpreted as transport-limited erosion in a uniform steady state. **I pointed out this more clearly at the end of Sect. 2 (lines 178–186)**.

"Because of the above assumption (uniform erosion rate), in the case of non-uniform uplift rate (and thus non-uniform erosion rate at steady state), the transported-limited model produces different final landscape compared to the detachment-limited model (Figure 1, middle and right panels). The proposed transport-limit model, assumed a uniform erosion, is unlikely suitable to study a non-steady-state (transient) landscape evolution or a non-uniform uplift scenario. Please argue against me if I am wrong.

Of course, the transport-limited end member is not suitable in bedrock mountain streams, and the detachment-limited end member not suitable for large parts of Earth's surface outside the mountain belts. We can conclude that we need combined models such as the one proposed by Davy & Lague (2009) in order to capture the majority of the rivers in the real world, but nothing more.

"The author mentioned that Yuan et al. (2019, JGR)s erosion-deposition model/method breaks down if the model approaches the transport-limited regime, which is not true. Yuan et al. (2019) mentioned in their article that '. . . , the iterative method is proven to converge unconditionally at least when $G \leq 1$, but we show experimentally that this method can also converge even if this condition is not satisfied', e.g., at $G = 10$ (their Figure 3a), which is in transport-limited regime for $G > 1$, a criteria estimated from various experimental and natural landscapes (Guerit et al., 2019, Geology)."

This is just due to slightly different terminologies. In my manuscript, I refer to the classical concept with detachment-limited and transport-limited erosion as end members that do, however, not occur in nature in this strict form. Anything in between would be called mixed channels then. In turn, you subdivide the entire range into a detachment-limited and a transport-limited range. **I clarified this by declaring detachment-limited erosion and transport-limited erosion as end members (lines 9, 17, 132, 134, 271, 272, 346, 364, 365, 387, 397, 399, 405, and 559).**

"L5: 'as the stream-power law is' change to 'of the stream-power law'."

I changed the wording of this sentence either **(lines 5–6)**.

"L7: 'as the established implicit solver for transport-limited erosion', should be detachment-limited erosion?"

Thanks! I could have read this 100 times without finding this mistake. **Fixed (line 8).**

"L23 and L25: 'sediment flux density' change to 'sediment flux per unit width'."

Good idea as flux density could be misinterpreted as flux per area. **I changed it throughout the paper (lines 26, 32, 115, 143, 151, 162, 359).**

"L67: A reference is needed for the upstream propagating velocity of erosion."

I would say it is too simple for a reference as it is just the velocity of advection in the advection equation. **I added a short explanation (line 84).**

"L72: 'but despite increasing computing capacities still important point' change to 'because increasing computing capacities is an important aspect in the landscape evolution modelling'."

I removed this phrase because everybody knows that computing capacities have increased (line 94).

"L78: Two 'n in $O(n)$ and Sn are confusing. Suggest to use '$O(N)$ and N is the number of nodes discretizing the landscape. Suggest to change throughout the manuscript."

Good point; I changed it throughout the paper (lines 96, 97, 254, 557).

"L134: Not easy to understand how to derive this equation (12) based on the above equations. Before the sentence, please write 'Combine equations (1) and (12)', ..."

Ok, although this is not the most difficult part of the paper. **I added the explanation (line 164).**

"L288-L291: It is better to list this computational time in Table 1."

Table 1 occurs much earlier in the paper. So I think that it would not be helpful to mention these values already there. Beyond this, it would make Table 1 more complicated as it refers to a single step so far.

Reviewer 2 (Wolfgang Schwanghart)

"164: $S_i(t)$ is actually not used later, at least in the immediate context (Eqs. 20-25). You may, however, replace the second term in the right-hand side of Eq. 20 with $S_i(t)$."

It is in fact used exactly the way you suggest since the direct form of Eq. (20) is $Q_i(t) = KA_i^{m+1}S_i$. But this is just the intermediate step from Eq. (11) to Eq. (20). In order to proceed to Eq. (21), however, I need the form of Eq. (20) used here. So the form of Eq. (20) with S_i would be an intermediate step that might help the readers, but it cannot be used as a simplification.

"What about upstream boundary conditions? I wrote the model in MATLAB as 1-D model and the uppermost node remains fixed (if no uplift is applied). I may have wrongly written the code, though."

The "natural" upstream boundary condition is no influx of sediment (homogeneous von-Neumann), which is implicitly defined by the condition that the direction of the sediment flux follows the flow direction of water. So grid cells that are not supplied with water from other cells also do not receive sediments. In the equations, this means that all sums over the donors (all $\sum_j$ terms) are empty. Then $\alpha_i = s_i = A_i$ and $\beta_i = s_i U_i = A_i U_i$ (Eq. 18),

$$Q_i(t) = \frac{A_i\left(H_i(t_0) - H_b(t)\right) + A_i U_i \delta t}{A_i \frac{d_i}{KA_i^{m+1}} + \delta t}$$

(Eq. 23), and then into Eq. (21):

$$H_i(t) = H_b(t) + \frac{H_i(t_0) - H_b(t) + U_i \delta t}{1 + \delta t \frac{KA_i^m}{d_i}}.$$

This expression is even the same as the fully implicit step for the detachment-limited model, and according the formulation in Sect. 2 detachment-limited erosion and transport-limited erosion must indeed be the same at those sites without donors. So I guess that it might be just a problem in your 1D implementation. **I added a short node on the boundary conditions (lines 217–219)**.

"In the end, the model is not fully 2D, as the scheme is solved in 1D on a network. This may lead to weird aggradational forms (linear ridges on flat topography) if too long time steps are applied. Can you comment on this? What is an appropriate time step length?"

Yes, this problem affects all models where the computation of the flow pattern is separated from the change in topography, so also the detachment-limited model. How severe it is and how much it limits the maximum δt, depends strongly on the considered situation. So it is even difficult to provide a rule of thumb for the maximum δt. **I added some remarks in the section about the limitations (lines 535–543).**

Reviewer 3

"One aspect of the method that I find interesting is that the implicit algorithm should apply equally well for one-dimensional diffusion problems, provided that the flow in question is always oriented in one direction. That might not be tremendously helpful for people who want to model diffusion, since we already have numerous well-known solution methods in 1D and 2D for that particular problem, but it does suggest a way to test the proposed scheme under transient conditions. The manuscript notes that investigating the temporal behavior turned out to be quite complex. Yet understanding temporal behavior is one of the reasons to use landscape evolution models in the first place. It is important to know something about the limits to accuracy and stability of a numerical scheme under transient conditions. I suggest therefore that the author try formulating a 1D, uni-directional diffusion problem and solving it with this implicit method for the transient case of a step change at one end of the domain. The analytical solution for that case is well known, so it seems like a good opportunity to test the properties of the proposed scheme under transient conditions. Basically, it would be a matter of having a 1D domain and a constant value of A."

Transient behavior is, of course, the heart of landform evolution modeling. However, catching up with what has already been done for the detachment-limited case just in one section of this paper is impossible. We know what solutions of the 1D diffusion equation look like, but the interesting aspect how disturbances propagate into tributaries where the diffusivity is lower than in the trunk stream. My recent Postdoc researcher has already started a study on the characteristic response time of catchments to changes in uplift or base level and which part the hillslopes play here. But this will be an own paper where I will not be first author.

"Such a test might also make it possible to identify constraints on time-step size. The manuscript notes that implicit methods allow arbitrarily large time steps. Yes thats true in principle, but arbitrarily large really just refers to stability. Two other considerations are: how does step size influence solution accuracy, and in particular for landscape models, to what extent does drainage network reorganization limit step size? These questions are undoubtedly hard or maybe impossible to answer in general, but some practical rules of thumb would be useful for those who wish to apply the algorithm in practice. So again I encourage the exploration of a transient case of simple 1D diffusion."

Formally, the error of the scheme is linear in δt, but this does not help much practically. The reorganization of the drainage network is indeed the limiting factor here. This problem affects all models where the computation of the flow pattern is separated from the change in topography, so also the detachment-limited model. How severe it is and how much it limits the maximum δt, depends strongly on the considered situation. So it is even difficult to provide a rule of thumb for the maximum δt. **I added some remarks in the section about the limitations (lines 535–543).**

"14 the word uplift has a long history of ambiguous usage among geoscientists. I recommend specifying uplift of crustal material relative to a given datum or something like that."

I think that it would not be a big problem for the readers in combination with the equation. Nevertheless, the is no argument against clarifying it, so **I added the suggested phrase (line 15)**.

"16 I suggest adding some references here for the benefit of readers who are just getting into the topic. I am not sure of the provenance of the term transport limited, but I think it appears in Carson and Kirkby (1972) in the context of hillslopes. For the landscape evolution context, Willgoose et al. (1990, Water Resources Research) might be a reasonable reference, though I do not remember whether they actually used this phrase. As far as I know, the term detachment limited was coined by Howard (1994, Water Resources Research)."

Admittedly, I am completely uncertain about the provenance of the two terms. I agree that it makes sense to mention the two papers from the early 1990s where the two concepts presumably occurred for the first term in the context of these types of models, so **I added the references (lines 19 and 30–31)**.

"26 Up to this point, you have not actually defined transport limited. This would be a good place to do so. I think of a transport-limited river reach as one in which the rate of bed erosion is limited by the ability of the flow to transport the eroded material downstream, rather than by the availability of potentially mobile sediment (feel free to use this wording if you like it)."

It did even not think that the term 'transport limited' requires a definition, but it makes sense and I like your suggested wording, so **I added it (lines 29–30)**.

"For what it is worth, in my view, the definition is actually fuzzier than we sometimes pretend: the ability of moving fluid to transport sediment depends very strongly on the size and density of sediment on the bed. There is no such thing as a transport capacity independent of bed sediment characteristics. Bed-load theory tells us that transport capacity depends on critical shear stress, which in turn depends on sediment size and density; suspended-load theory tells us that sediment concentration depends on near-bed sediment concentration and on settling velocity, both of which also depend on size and density. But for purposes of this paper, the only real practical implication of this observation is that one should be cautious in using the phrase transport capacity. "

I fully agree. The consequence is that the parameter K in my formulation is a lumped parameter that also depends on the characteristics of the material coming from the upstream area. **A little remark in this direction now occurs in line 417–419**.

“31-33 Second-order derivatives only appear if q is a function of topographic gradient. Suggest adding wording to clarify this, e.g., Because q is a function of topographic gradient, eq (1) contains... ”

I took it out of the parentheses and explained it a bit more clearly now (lines 39–40).

“35 Change In the last years to In recent years ”

Corrected (line 43).

“36 there seems to be a trend to – I also share the impression that use of a detachment-limited stream erosion model in landscape evolution studies is common, but whether there has been a trend in that direction is harder to say. There are situations in which a transport-limited model is suitable, and plenty of literature on such models (e.g., Wickert and Schildgen, 2019; and a great deal of the work by Greg Hancock, Tom Coulthard, and colleagues). Suggest simply asserting that detachment limited is a common or popular choice.”

This indeed sounds better, so **I changed the wording (line 44).**

“40 Of the three suggested reasons for the widespread use of detachment-limited discharge-slope models, I think the second two are really the important ones. The first might be a bit misleading to readers, because any of the three flavors of model discussed in this paper can be related to a power-law slope-area relationship. As far as I know, the link between erosion/transport and slope-area was actually first identified in a transport-limited context. If I recall right, Howard (1980 in Thresholds in Geomorphology) articulated a slope-area relation based on a variety of different transport formulas, and Howard and Kerby (1983) followed up with a field-based study. Then Willgoose et al. (1991) and Willgoose (1994) really hit home the slope-area relation in a transport-limited context. So having a link with Flints law is not unique to the detachment-limited formulation. The solution I suggest is just to add a sentence, maybe after the sentence following eq (6), to the effect that transport-limited and other types of erosion law can also be linked to Flints law (references), but the relationship is especially simple for the area-slope erosion law in eq (6).”

This is basically true, but nevertheless the transport-limited approach seems not be used in such a simple form as the detachment-limited approach nowadays. **I adjusted the wording so that it no longer implies that the relationship to Hack’s findings is clearer, but only simpler for the transport-limited model (lines 49–50) and changed in the abstract accordingly (lines 5–6). In addition, I mentioned that Willgoose (1991) already brought the transport-limited model into the context of Hack’s findings (lines 76–81)**.

"49 has become some kind of paradigm – I think I understand what you mean here, but as written it is a vague statement (what exactly constitutes a paradigm? what kind of paradigm?). Better I think to leave this comment out."

I replaced it by "a common choice" (line 59).

"56 little is known – this statement is a bit unfair to researchers who have tried to pin it down. Suggest softening to something like the effective value of n is less well known. You could also add something like: some studies suggest a linear scaling (REFS), some sub-linear (REFS), and some super-linear (REFS)."

I know about some fields where several highly reputed scientists spent much work, and there is still very limited knowledge. Estimating the value of n is just extremely difficult and susceptible to systematic errors. Personally, I do not trust in any of the estimates of n from the literature very much, but discussing this would be a different paper. **I used a softer wording now (lines 66–67) and referred to Lague (2014) for an overview (lines 70–71). In addition, I discussed one of the potential sources of systematic errors at the end of Sect. 4 now (lines 318–344)**.

"62 There are quite a few other papers that report estimates of K values, which could be cited here. I guess the e.g. is meant to say there are more papers than I feel like bothering to list here, but if you want a starting point, try these two. I guess thats ok, but you are likely to annoy the authors of the ones you left out. An alternative would be to find a recent paper or two that reports K values and is reasonably comprehensive in its referencing, and cite as So-and-so, 20xx, and references therein."

The list was even not thought to be a starting point for studies about K in general. The point is that the term "erodibility" could be misinterpreted as a property of the rock alone, while it is a lumped parameter. These two references particularly refer to the dependence on climate, and the "e.g." is due to that fact that I am not sure whether there are also older references addressing the dependence on climate.

"76 The wording is a bit awkward here; suggest leaving out despite increasing computing... (we all know computers have gotten faster)."

True, **I removed this phrase (line 94)**.

"87 models treat "

Thanks! **Fixed (line 105)**.

"100 confusing because you would choose either (2) or (8); how about (1) and either (2) or (8) "

No, we would indeed need the 3 equations in order to obtain a system of 2 partial differential equations for the variables H and q in this case. However, as this part is indeed a bit complicated, **I moved it to the new Sect. 5 and explained it in more detail there (lines 357–359)**.

"109 the upstream"

Fixed (line 128).

"eq (11) and preceding text: the way this is written seems to suggest that approaching the problem from the question of how much sediment would you get from eq (6) is a requisite for deriving the method that follows. Actually, there are at least two other pathways that I can think of. I think you are more likely to sell the approach more effectively if you point out that there are several lines of evidence to support the hypothesis that the long-term sediment flux should depend on slope and drainage area. You have ar- ticulated one of them, but it seems to me it is subject to the criticism that you are using a detachment-limited concept (eq 6) to derive a transport-limited model. An alternative would be to state that previous studies have shown that sediment-transport formulas can be cast in the form of an area-slope power expression, and cite some references. You could also lean on Davy and Lague (2009) here, because when you combine their expression with a unit-stream-power detachment rate and $Q_w \propto A$ (Q_w being water discharge), you end up with a transport capacity (if I recall right) that looks like $A^{\frac{3}{2}}S$ (more generally, $A^{m+1}S^n$). I think it is fair to say that there is uncertainty in the literature over how best to express transport capacity in models of stream profile evolution or landscape evolution. Some have $Q \propto Q_w S$, some (e.g., Willgoose, Howard, based on the empirical Einstein-Brown expression) have $Q \propto q_w^2 S^2$, and some include a transport threshold. Key point for your purposes is that $Q = KA^{m+1}S$ falls within the span of proposed laws."

It indeed reads a bit as if this section started from the detachment-limited model and measured the resulting sediment yield. However, I rather thought of the generic model based on Hack's findings, i.e., on a uniform effective erosion rate that yields river profiles with constant concavity and steepness indices. At this level, it just describes an erosion rate without regard to any mechanism. **I tried to point this out more clearly in lines 141–149, mentioned the alternative approach using more physical principles in the introduction (lines 76–81), and return to this point in lines 168–171.** However, I still prefer the empirical starting point here in order to point out that both interpretations of Hack's findings are somehow straightforward.

"130 change which to that (introduces a restrictive clause)"

Thanks! **Fixed (line 158)**.

"132-3 I do not understand this comment about Voronoi polygons. Normally in a finite-volume solution, you would integrate flux density over the width of a cell face, whether it is a square or a Voronoi polygon or some other shape. Using Q instead of q, with an implied sub-grid-scale channel width (I suppose), you do not need to do this integration; but that is true regardless of the shape of your cells. Is your point that the discrete representation in eq 12 works in principle for any grid mesh, regular or irregular? Consider removing this statement, as it seems like a bit of a distraction."

You are right, the Voronoi condition is not essential here, although of advantage for the accuracy. **I formulated it for a general finite-volume discretization now (lines 160–163)**.

"137-8 See comment above about transport laws. Equations like (11) have been frequently used in the literature. In particular, in the work of Willgoose et al. (1991a,b,c) and subsequently, the slope-area relationship is used to estimate parameters for a transport law. If there is something very specific about eq (11) that you think is unique, then that should be pointed out. Otherwise, the statement carries the implication that transport laws have never before been derived from slope-area analysis, which is not correct."

I clarified this point (lines 141–149 and 168–172, see also the above comment). But beyond this, I think the simple formulation given in Eq. (11) is new, and this its advantage is explained in lines 173–180.

"144-5 I believe it is more than a matter of terminology. It is rather a matter of dimensionality. If you suppose $m = 1/2$, then the erodibility has dimensions of inverse time, whereas the transport coefficient has dimensions of length^2/time."

No, both have the dimension of inverse time and even the same value under identical conditions (same river profile at the same erosion rate).

"153-157 Consider adding some more explanatory text here. At first glance, eq (15) looks like a Taylor expansion to first order. But if I am following this correctly, actually Q_{0i} includes the value of Q_i at t_0 plus the partial derivative of Q_i with respect to H_i times the change in H_i during one time step. That's a clever idea, and is consistent with the explanation on line 156, but it took me some time to work it out. Other readers might similarly misinterpret Q_{0i} on a first look, and yet its definition is really key to whole scheme. Suggest devoting a full sentence or so to pointing out the definition and importance of it.

The linear dependence of all properties (height, flux) on the base level, i.e., that Eq. (15) is exact, is indeed the key to understanding the numerical scheme. **I added a detailed explanation including the new Fig. 1 where it can be recognized that it is not just a first-order Taylor approximation, but an exact relation (lines 201–207).**

"177 The challenge for readers is that the donor information is buried in the definitions of alpha and beta. Suggest adding, after the word donors, (because α and β depend on donors Q and Q', respectively)."

Ok, seems to be helpful, **so I added something like this (lines 234–235)**.

"187, 189 - reference to a recursive implementation is vague. Suggest referring to a published algorithm(s) for sorting by downstream order."

To my knowledge there is no sorting algorithm that achieves linear complexity ($O(N)$) on average, so sorting the nodes would formally destroy the linear complexity for both transport-limited and detachment-limited erosion. Provided that a programming language that supports recursion is used, a recursive scheme is more efficient than sorting. **I briefly explained how the recursive schemes have to be designed here (lines 245–249 and 250–251)**.

"198-200 Please document somehow the specifications for the performance tests: for example, the number of iterations were run for each case.

Seriously? The values are, of course, obtained from several runs (here with 100 and 1000 time steps) and checked for consistency. However, we know that the uncertainty is primarily not due to the process of measuring, but to the details of the implementation of the individual functions. So **I added one short remark (lines 259–260)**.

"207-8 With all due respect, I think this is a missed opportunity. As noted above, I suggest trying a solution with one row of grid nodes (so, strictly one dimensional) and a uniform drainage area. Then it reduces to a linear diffusion equation, which you could compare with the transient analytical solution for diffusion given a step change at one boundary."

I am quite sure that this opportunity does not run away so soon, and there is already some ongoing work in this direction. However, the 1-D version is not interesting enough and thus not really an option.

"213 Please explain the rationale for increasing δt over time."

Ok, **I explained the scheme a bit more in detail, although this required some reordering (lines 279–292)**.

"217-8 To avoid potential confusion, it would be useful to clarify that the two models are NOT equivalent, but rather their steady state solutions have the same slope-area relationship. Either give the predicted slope-area equivalence, or quote a reference that does (or both)."

Good point! **I discussed the condition under which the models are equivalent and how they differ at the end of Sect. 2 now (lines 178–186)**.

"223 and following: The tent-shaped uplift pattern is a clever test. I think the example would be easier to follow if you did two things. First, before referring to the results (figure 2), explain why you are using this tent-shaped uplift and what differences you expect to see between the two models. That way, the reader knows what to look for in Figures 2 and 3. Second, it would be very helpful to provide an analytical solution for the two models. You could simply use Hacks law to relate drainage area to distance (I would just make the exponent 2 for simplicity). Plot the predicted longitudinal stream profiles with a tent-shaped uplift pattern for each model, in chi space (you could do linear space too). That way there is a clear expectation for Figure 3 (actually, you could simply add the analytical profiles to Figure 3). If I have done the math right, the two profiles should be defined by

$$dH/dx = (u0(Lx)/K)x^{-hm}$$
$$dH/dx = (u0/K)x^{-h(m+1)} \int_0^x (L-x)dx$$

where u_0 is the uplift rate at the ridgeline, h is the Hack exponent, and L is the domain half-width. So, should be possible to plot these as analytical expectations."

A good idea in principle, but it runs into problems with the contribution of tributaries for the transport-limited model. In contrast to the catchment sizes, the sediment fluxes do not obey Hack's law. As an example, all single-pixel catchments are located at the ridge and thus exposed to the maximum uplift in the 1D formulation using Hack's law. In the network, however, they are distributed over the entire catchment. So they are exposed to lower uplift on average and thus provide less sediment. I recently worked on a similar problem in the context of glacial erosion (Prasicek et al., EPSL, 2020, 0.1016/j.epsl.2020.116350), and it is more complicated that it seems first.

"251 ϕ and ψ seem to be parameters rather than functions."

This depends on the point of view. If we consider the catchment size A as a variable, ϕ and ψ are functions. **I clarified this by writing $\phi(A)$ and $\psi(A)$ at several locations where it makes the terms not too cumbersome**.

"254 Davy and Lague deserve much credit for introducing this formulation in the landscape evolution context, and showing that it relates to the earlier Beaumont model except that the length scale varies with unit discharge. For the record, similar formulations with erosion/entrainment and deposition terms seem to be widely used in the sedimentation engineering and soil erosion communities."

Good point, but I must admit that I did not work on these fields since the end of the 1990s, and I am not familiar with the recent literature.

"eq (34) I like this alternative expression of phi. Presumably it would simplify calibration by removing a built-in correlation between the two parameters."

I even used the model like this for several years in such way that K defines the steepness of equilibrium profiles and G let us more between detachment-limited and transport-limited. The Associate Editor also suggested to discuss this class of models in more detail, and **new part additionally contains one more alternative formulation with a different set of parameters (lines 396–419)**.

"284-5 Would not $G \to \infty$ lead to $Q_i \to 0$ by eq (31)?

Indeed! This is another argument why the Davy-Lague model must be rescaled in order to have a well-defined transition to the transport-limited end member. **This is discussed more thoroughly now in Sect. 5 (lines 365–368)**.

"286 missing to "

Fixed (line 458).

"287 extra of "

Fixed (line 459).

"294 some kind of is a bit vague. Suggest re-wording to be more precise."

This sentence has vanished by some restructuring the sections anyway (line 475).

"295-6 Can you articulate what process(es) this kind of formulation is meant to represent? Is the idea that some of the material is so fine-grained that it will not end up being deposited until it reaches the ocean or some kind of closed basin? I wonder whether an alternative would be to build this into dQ/dH."

This was exactly the idea behind it. **I added a note on the idea (lines 478–479)**.

"303-4 This statement is not clear to me. From the references cited, I guess that by scaling problem you mean the classic problem of grid-size scaling. Yet that wasn't mentioned as an issue with the prior models (transport limited and linear decline), so why is it more of an issue with equation 35 than with, say, equation 26?"

Admittedly, the explanation is very short. The reason why it may occur here, but not in the previous consideration is breaking the sediment balance (direct excavation) along linear river segments (line 487). It may become clear to the readers if they proceed to the reference Hergarten (2020a). However, it is just thought as a warning for those who ever get to the point where they want to implement such a model.

"317-325 Are you suggesting to solve for diffusive flux in the flow directions using the implicit scheme, and the other directions using some other scheme? How would you avoid double-counting the fluxes in the cardinal flow directions? Overall, I think the sketch presented here for handling diffusion is not really convincing. I would recommend either deleting it, or expanding it to really demonstrate how it would work."

Exactly! I thought it would be not a big challenge to follow the idea, but **I explained it in more detail now (lines 504–519)**.

Associate Editor (Jean Braun)

"I think the author could provide a more quantitative comparison between the results obtained with this new method and those obtained by solving the Davy and Lague (2009) approach directly (i.e. without using the flux divergence formulation) as done by Yuan et al (2019). I have rapidly coded the two approaches in 1D and easily showed that the improvement in speed is a strong function of G (for $G \geq 1$).

You are right that the improvement in performance compared to the iterative implementation of Yuan et al. (2019) rapidly increases with increasing G for $G > 1$ as the convergence of the iterative scheme slows down. The reason why I did not consider this in such detail is that the estimates of Yuan et al. (2019) might even be too pessimistic. I used basically the same approach for some years, but with a fixed number of 2 iterations (the minimum number of flow directions change), and did not care for convergence. I also work reasonably well, in particular if we take into account that changes in flow directions require a limitation of δt anyway. That was the reason why I stayed on a rather qualitative level. i.e., focusing on the advantage that we do not have to take care of anything when using the direct scheme. Nevertheless, **I introduced some numbers on the gain obtained from comparing the results of Yuan et al. (2019) and Guerit et al. (2019)**.

"Additionally to the suggestions made by the reviewers, I would like the author to provide a more structured explanation of the procedure used in the solution of what the author calls the 'linear decline model'. I note that the author's implementation is based on equations [1]+[26] (I use square brackets to indicate equation numbers from the manuscript and parentheses to indicate equation numbers from this comment):

$$\frac{\partial H}{\partial t} = U - \phi S + \psi Q \quad (1)$$

and its discretised form [28]; but, of course, it also uses equation [17], which is equivalent to [16], itself a discretised form of equation:

$$\frac{\partial H}{\partial t} = U \mathrm{div} q \quad (2)$$

This leads me to conclude that the author uses two evolution equations simultaneously at every point of the landscape. Combining them also suggests that:

$$\mathrm{div} q = \phi S - \psi Q \quad (3)$$

This may explain why the method is hybrid, depending on the value of the ψ parameter (purely advective if $\psi = 0$ and divergence based (or diffusive) for large values of ψ), as pointed out by the author. However, it is not clear to me how to connect all these points together. I would appreciate if the authors could help me (and other readers) clarify this point with a more structured presentation of the basic partial differential equation(s?) that are solved in the linear decline model algorithm.

Exactly! **As the main change to the manuscript, I devoted an own section to the linear decline model now (new Sect. 5). This section starts from the theoretical point of view – two coupled partial differential equations and their "usual" formulation as a single integro-differential equation. Then it addresses the limiting cases where it turns into a single differential equation, however, of different types for the detachment-limited and transport-limited end members. As a little extension, I introduced a formulation in terms of two different parameters K_d and K_t, which can be interpreted as shared stream power.** I hope you find this section useful.

In addition, I extended the results section by Fig. 5 (lines 318–344) and some text. I did this mainly because Reviewer 3 brought the exponent n into play. In the recent literature, there seems to be some trend towards rather high values of n. Taking into account the results from the simple tent-shaped uplift pattern for transport-limited erosion, I am wary about these large values of n, so I discussed the results of this example in this context as some kind of warning. Maybe you could take a look at the reasoning and let me know whether you find it useful. It is not an essential part and could easily be skipped.

Best regards,

Stefan Hergarten

Transport-limited fluvial erosion – simple formulation and efficient numerical treatment

Stefan Hergarten[1]

[1]Institut für Geo- und Umweltnaturwissenschaften, Albertstr. 23B, 79104 Freiburg, Germany

Correspondence: Stefan Hergarten
(stefan.hergarten@geologie.uni-freiburg.de)

Abstract. Most of the recent studies modeling fluvial erosion in the context of tectonic geomorphology focus on the detachment-limited regime. One reason for this simplification is the ~~direct~~ simple relationship of the constitutive law used here – often called stream-power law – to empirical results on longitudinal river profiles. Another, not less important reason lies in the numerical effort that is much higher for transport-limited models than for detachment-limited models. This study proposes a ~~simple~~ formulation of transport-limited erosion ~~that is as close~~ where the relationship to empirical results on river profiles ~~as~~ is almost as simple as it is for the stream-power law~~is~~. As a central point, a direct solver for the fully implicit scheme is presented. This solver requires no iteration for the linear version of the model, allows for arbitrarily large time increments, and is almost as efficient as the established implicit solver for ~~transport-limited~~ detachment-limited erosion. The numerical scheme can also be applied to linear ~~models between the two extremes~~ hybrid models that cover the range between the two end members of detachment-limited and transport-limited erosion.

1 Introduction

Rivers play a major if not dominant part in large-scale landform evolution. If horizontal displacement of the crust is not taken into account, models describing the evolution of a topography $H(x_1, x_2, t)$ are typically written in the form

$$\frac{\partial H}{\partial t} = U - E, \tag{1}$$

where U and E are ~~uplift rate and erosion rate~~the rates of uplift of crustal material relative to a given datum and of erosion, respectively.

Two ~~limiting cases~~ end members – detachment-limited and transport-limited erosion – are widely considered in the context of fluvial landform evolution. ~~For~~ The term "detachment-limited erosion~~, it is assumed~~" was presumably coined by Howard (1994). The idea behind this concept is that all particles entrained by the river are immediately removed from the system. The erosion rate E can be considered as a function of local properties at each point. In the simplest approach, these are catchment size and channel slope (slope in direction of steepest descent), while all other influences are subsumed in a lumped parameter often called erodibility.

In all scenarios other than the detachment-limited case, a sediment balance must be considered. If no material is directly removed, the erosion rate is

$$E = \mathrm{div}\boldsymbol{q}, \tag{2}$$

where $\boldsymbol{q}$ is the sediment flux density per unit width (volume per time and cross section length) and div the 2-D divergence operator. It is usually assumed that $\boldsymbol{q}$ follows the direction of the channel slope, so only its absolute value q varies between different models.

The concept of transport-limited erosion assumes that the rate of bed erosion is limited by the ability of the flow to transport the eroded material, rather than by the availability of potentially mobile sediment. The implementation of this concept in fluvial landform evolution models presumably dates back to Willgoose et al. (1991b). Transport-limited models directly define the sediment flux density per unit width q instead of the erosion rate E at each point as a function of local properties such as catchment size and channel slope.

Mathematically, both concepts differ fundamentally. Equation (1) only involves derivatives of first order with regard to time and with regard to the spatial coordinates (arising from the channel slope) in the detachment-limited scenario. So it is a hyperbolic differential equation of the advection type. Propagation of information in one direction only – upstream here – is a characteristic property of this type. Anything that happens at a given point and a given time only affects the region upstream of this point in the future. In contrast, Eq. (1) contains spatial derivatives of second-order in the transport-limited regime (from the channel slope and from the divergence operator)since $\boldsymbol{q}$ inside the divergence operator depends on the channel slope. Equation (1) combined with Eq. (2) is a parabolic differential equation of the diffusion type then, where information propagates in both upstream and downstream direction.

Several comprehensive numerical models of fluvial landform evolution have been developed since the 1990s. All models reviewed by Coulthard (2001), Willgoose (2005), and van der Beek (2013) involve a sediment balance. In the last recent years, however, there seems to be a trend to using the detachment-limited model has become a popular choice, although the idea that all particles are immediately excavated is limited has been questioned (e.g., Turowski, 2012). All types of bedload transport are obviously not captured by this concept. Nevertheless, even some recent studies using models that are able to simulate sediment transport focus on the detachment-limited case (e.g., Duvall and Tucker, 2015; Theodoratos et al., 2018; Eizenhöfer et al., 2019).

At least three aspects make the detachment-limited approach appealing. First, there is a close relationship to old the relationship to empirical studies of longitudinal channel profiles is particularly simple here. Hack (1957) observed a power-law relationship between channel slope S and upstream catchment size A in several rivers. This relationship is nowaday often called Flint's law (Flint, 1974) and written in the form

$$S = k_s A^{-\theta}, \tag{3}$$

where θ is the concavity index and k_s the steepness index. Assuming that Eq. (3) is the fingerprint of a spatially constant erosion rate under uniform conditions, it can be assumed that

$$E = f(k_s) = f(A^\theta S), \tag{4}$$

where f is an arbitrary function. ~~Assuming a~~ A power-law function ~~,~~

$$f(k_s) = K k_s^n = K(A^\theta S)^n, \tag{5}$$

where the parameter K is denoted erodibility, ~~has become some kind of paradigm~~ is a common choice in this context. The fluvial erosion rate is often written in the form

$$E = K A^m S^n \tag{6}$$

with $m = \theta n$. Equation (6) is often called stream-power law and its combination with Eq. 1 stream-power incision model since it can be interpreted in terms of energy dissipation of the water per channel bed area if an empirical relationship between channel width and catchment size is used (e.g., Whipple and Tucker, 1999).

The concavity index $\theta = \frac{m}{n}$ appears to be well constrained, so most modeling studies either use the value $\theta = 0.5$ originally found by Hack (1957) or a slightly lower reference value $\theta = 0.45$ (e.g., Whipple et al., 2013; Lague, 2014). In turn, ~~little is known about the~~ the value of the exponent n is less well constrained since it cannot be ~~constrained~~ determined from the shape of equilibrium profiles under uniform conditions. The model is linear with regard to H (if the flow pattern is given) for $n = 1$, which simplifies both theoretical considerations and the numerical implementation. Thus, the ~~lack of clear knowledge about~~ remaining uncertainty in the effective value of n often serves as a reason for choosing $n = 1$, although, e.g., the results compiled by Lague (2014) rather suggest $n > 1$. If θ is well constrained and $n = 1$ is accepted as a convenient choice, the erodibility K remains as the only parameter. It is a lumped parameter subsuming all influences on erosion other than channel slope and catchment size~~, so~~ . So it is not only a property of the rock, but also depends on climate in a nontrivial way (e.g., Ferrier et al., 2013; Harel et al., 2016). However, it just defines how steep rivers will become at a given uplift rate, so reasonable values can be found, e.g., by analyzing river profiles at situations where estimates of the uplift rate are available.

~~The~~ Constitutive laws based on power-law relations, however, have not been employed only in detachment-limited models. Even the earliest numerical model of transport-limited erosion (Willgoose et al., 1991b) used a power law for the sediment flux density based on physical relations for the shear stress at the bed. The empirical results on real rivers represented by Eq. (3) were also used to constrain the parameter values before the detachment-limited concept became popular (Willgoose et al., 1991a). However, the transport-limited approach never reached the simplicity of the detachment-limited approach with regard to the small number of parameters and their quite direct relation to the properties of real river profiles.

The simplicity of the differential equation itself serves as a second argument in favor of the detachment-limited approach. In the linear case ($n = 1$), Eq. (1) combined with Eq. (6) can be solved analytically for any given uplift pattern and history. ~~Disturbances~~ The term KA^θ defines the velocity of advection then, so disturbances propagate in upstream direction at a ~~velocity KA^θ~~ this velocity. The treatment can be simplified by the χ transform introduced by Perron and Royden (2013). It transforms the upstream coordinate x to a new coordinate

$$\chi = \int \left(\frac{A(x)}{A_0} \right)^{-\theta} dx, \tag{7}$$

where A_0 is an arbitrary reference catchment size and the integration starts from an arbitrary reference point. This transformation eliminates the inherent curvature of river profiles arising from the decrease of catchment size in upstream direction, so equilibrium profiles under spatially uniform conditions turn into straight lines. The solutions of this equation and their potential for unraveling the uplift and erosion history were investigated by Royden and Perron (2013), and a formal inversion procedure for the linear case ($n = 1$) was presented by Goren et al. (2014). So the detachment-limited model can be reconciled with real river profiles not only under steady-state conditions, but also in the context of temporal changes.

As a third ~~, but despite increasing computing capacities still important~~ point, detachment-limited erosion can be implemented in numerical models more efficiently than transport-limited erosion. Here, even a fully implicit scheme that allows for arbitrary time increments with linear time complexity, also known as ~~$O(n)$~~$O(N)$, is available. This means that the computing effort increases only linearly with the total number of nodes N. The scheme was introduced in the context of fluvial erosion by Hergarten and Neugebauer (2001), described in detail for $n = 1$ and $n = 2$ by Hergarten (2002), and made popular by Braun and Willett (2013).

So far there is no comparable implementation for transport-limited erosion. As mentioned above, transport-limited erosion corresponds to a diffusion-type equation. The challenge is that the diffusivity depends on the catchment size and thus varies over several orders of magnitude. Multigrid methods (e.g., Hackbusch, 1985) are still the only schemes for the diffusion equation in more than one dimension with linear time complexity. However, convergence breaks down if the diffusivity varies by some orders of magnitude, so multigrid methods have not been applied in the context of fluvial erosion. So far none of the existing landform evolution model ~~treats~~ treat the transport-limited case with a fully implicit scheme that allows for arbitrarily large time increments.

The advantage of the detachment-limited model concerning the numerical complexity persists if explicit schemes are used here, too. The main reason for using explicit schemes for detachment-limited erosion is the artificial smoothing of knickpoints by the implicit discretization, while explicit schemes that preserve the shape of knickpoints better are available. A comparison was given by Campforts et al. (2017). As already pointed out by Howard (1994), explicit schemes for the transport-limited case typically require 3 to 4 orders of magnitude shorter time steps than for the detachment-limited case.

Howard (1994) already developed an approximation that makes the explicit scheme for the transport term numerically more stable. Kooi and Beaumont (1994) proposed an approach that increases stability and also allows for a physical interpretation, often called undercapacity model or – in a more general context – linear decline model (Whipple and Tucker, 2002). It defines an equilibrium flux ~~density~~ per unit width q_e from local properties (channel slope, catchment size, ...) and assumes that the erosion rate is

$$E = \frac{q_e - q}{l}. \tag{8}$$

The parameter l defines a length scale and can be seen as inertia of sediment detachment and deposition against changes in fluvial conditions. ~~The model consisting of Eqs. (1), (2), and (8) can be treated numerically by converting Eq. (2) to an integral equation based on the relation~~

$$Q = \int E \, dA,$$

~~where Q is the sediment flux (not flux density) and the integral extends over the upstream catchment of the considered point. Converting Q to a flux density and inserting it into Eq. (8) yields an integro-differential equation for the surface height H.~~

An alternative physical interpretation of the linear decline model was developed by Davy and Lague (2009). The detachment-limited model (Eq. 6) was extended by a sediment deposition term proportional to the actual sediment flux. As a main point, Davy and Lague (2009) found an expression for the rate of deposition that keeps equilibrium river profiles consistent with Eq. (3), which is not the case for the original undercapacity model (Whipple and Tucker, 2002).

Yuan et al. (2019) implemented an implicit numerical scheme for this model based on a Gauss-Seidel iteration in the upstream direction. The ~~convergence rate of the iteration~~ rate of convergence was found to be independent of the size of the grid, so the scheme is indeed of linear time complexity. ~~The rate of convergence, however, decreases~~ However, the convergence slows down strongly for faster deposition~~and breaks down if the model approaches the~~, i.e., when approaching the transport-limited ~~regime~~end member. ~~It~~ The scheme of Yuan et al. (2019) is therefore presumably the most efficient implementation of sediment transport in large-scale fluvial erosion models, but ~~it still cannot come close to~~ achieves its full power only if we do not come too close to the transport-limited ~~regime~~end member.

In the following section, a formulation of transport-limited erosion is proposed that can be directly reconciled with the concept of the erodibility. Then, Sect. 3 presents a fully implicit, direct scheme for solving the equation numerically. After presenting a numerical example in Sect. 4, Sect. 5 provides a discussion several versions of the linear decline model and an extension of the numerical scheme for this class of models.

2 Simple formulation of transport-limited erosion

Let us start from the interpretation of Hack's empirical relation (Eq. 3) as the fingerprint of uniform erosion under spatially constant conditions~~. Then the the sediment flux at each point of a river~~, regardless of the mechanism of erosion. This implies that the erosion rate is a function of the steepness index (Eq. ~~9)is the~~ 4). Then the sediment flux Q (volume per time, not per unit width) through any cross section of a river is

$$Q = \int E \, dA, \tag{9}$$

where the integral extends over the upstream catchment. For uniform erosion, the integral reduces to the product of the erosion rate and the catchment size,

$$Q = AE = Af(A^\theta S). \tag{10}$$

~~where f is the same function used for the detachment-limited model~~ If a power-law function (Eq. ~~4). If the stream-power approach (Eq.~~ 6) is used in analogy to the detachment-limited model, the sediment flux ~~is~~ becomes

$$Q = KA^{m+1}S^n. \tag{11}$$

In contrast to the more common formalism based on the flux ~~density~~ per unit width q (Eq. 2), these relations use the total sediment flux Q (volume per time) passing the entire cross section of a channel segment. This total flux cannot be inserted formally into the divergence operator in Eq. (2) to form a continuous differential equation. Practically, however, this is not a problem for a discrete channel network. If any pixel of the considered topography has a unique drainage direction towards a single neighbor and sediment transport follows flow direction, the respective discrete version of the divergence operator at the node i is

$$\mathrm{div}\boldsymbol{q}_i = \frac{Q_i - \sum_j Q_j}{s_i}, \tag{12}$$

where Q_i is the flux from the node i to its flow target. The sum extends over all neighbors ~~which deliver their sediment~~ that deliver their discharge und thus their sediment flux to the node i, called donors in the following. Finally, s_i is the area of the considered node, i.e., the pixel size for a regular mesh or the area of the respective ~~Voronoi polygon for a triangulated irregular network (TIN). On a TIN, this formulation is practically even simpler than the version based on the flux density because the lengths of the~~ cell in a general finite-volume discretization. As the model describes the total sediment flux and not flux per unit width, an integration over the edges of the ~~Voronoi polygons are not needed.~~ cell is not necessary.

~~The~~ Inserting Eqs. (2) and (12) into Eq. (1) then yields the simplest form of a transport-limited fluvial erosion model~~then reads~~,

$$s_i \frac{\partial H_i}{\partial t} = s_i U_i - Q_i + \sum_j Q_j, \tag{13}$$

where Q_i is defined by Eq. (10) or Eq. (11).

As mentioned above, using power-law functions for sediment transport is not new. In combination with empirical relations for the channel width, physically-based relations for the sediment flux density (e.g., Willgoose et al., 1991b) support the hypothesis of a power-law dependence of Q on A and S (Eq. 11). However, the relations where never written in such a simple form as in Eq. (11) with parameters that are related so closely to the concepts of concavity index and steepness index (Eq. 3). Equation (11) was ~~already~~ discussed in the literature (e.g., Whipple and Tucker, 2002) in the context of equilibrium river profiles, but apparently never used directly for defining a transport-limited erosion model. In view of Hack's findings this is, however, as straightforward as describing detachment-limited erosion by Eq. (4) or Eq. (6). Even the ~~meaning~~ physical unit of the erodibility K is the same in both models, ~~so that estimates~~ and the same values of K ~~inferred from measurements can be used the same way in both models. The only difference is that K is a catchment-wide erodibility (obtained by averaging the erosion rates over~~ yield the same erosion rate at the same topography for spatially uniform erosion.

The two models are, however, not equivalent for non-uniform erosion. According to Eqs. (4) and (5), the steepness index k_s directly reflects the erosion rate at the considered point in the form

$$K k_s^n = E \tag{14}$$

for the detachment-limited model. In turn, Eq. (11) of the transport-limited model can be combined with Eq. (9) to

$$K k_s^n = \frac{Q}{A} = \frac{1}{A} \int E \, dA. \tag{15}$$

This relation is basically the same as Eq. (14) except that the right-hand side is the mean erosion rate of the upstream catchment ~~), while it is a local property~~ instead of the local erosion rate at the considered point. So channel steepness directly reflects the local erosion rate in the detachment-limited model~~. However, it should be kept in mind that~~ , but the mean erosion rate of the catchment for the transport-limited model.

The same holds for the interpretation of the erodibility K ~~also carries information about the entire~~ . In the detachment-limited model, it describes how much material is eroded at the considered location for a given steepness index. In turn, it describes how much material is eroded on average in the upstream catchment in the ~~detachment-limited modelif precipitation varies within the catchment~~transport-limited model. From a process-oriented point of view, K would rather be considered a transport coefficient than an erodibility here. However, this is just a matter of terminology~~where the term erodibility has already been used~~.

3 A fully implicit numerical algorithm for transport-limited erosion

The model proposed in the previous section can be treated with an efficient, fully implicit numerical scheme in the linear case ($n = 1$). The reason why this is possible in contrast to the 2-D diffusion equation lies in the tree structure of the flow and sediment transport pattern.

The fully implicit discretization of Eq. (13) reads

$$s_i \frac{H_i(t) - H_i(t_0)}{\delta t} = s_i U_i - Q_i(t) + \sum_j Q_j(t), \tag{16}$$

where the time step extends from t_0 to t and $\delta t = t - t_0$. The solution at t_0 is known, and the solution at t is computed.

Let the node b be the flow target of the node i, so H_b serves as a base level for the node i. As the entire problem is linear, the height H_i responds linearly to base level changes. Figure 1 illustrates this behavior in a simple numerical example of a river (solid lines) with one tributary (dashed lines). The initial state ($t = 0$, blue) was a steady state under constant uplift. The red curves ($t = 1$) show the result of an implicit time step without uplift and with the same base level ($H_b = 0$) as for $t = 0$, while the orange curves correspond to different base levels H_b. The four red and orange curves of each river are equidistant at each point x for equal increments in H_b, so the change in height $H_i(t)$ due to changes in base level is proportional to the change in base level $H_b(t)$.

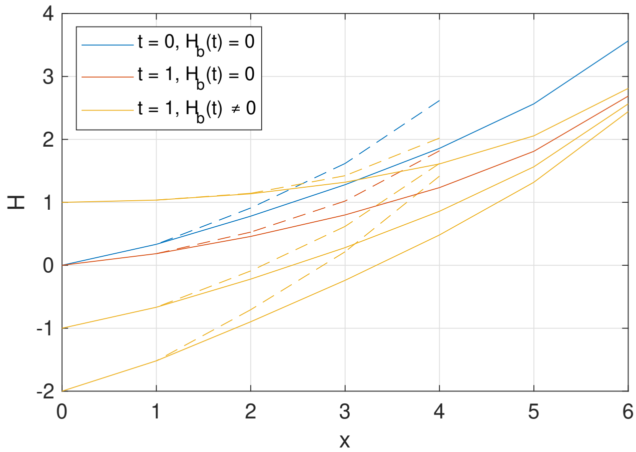

Figure 1. River profiles obtained from one implicit time step, where all parameters (K, δt, grid spacing) are set to unity. The blue line describes a steady state with $U = 1$, and it is assumed that $U = 0$ for $t > 0$.

Due to the linearity, the sediment flux Q_i to the node b also responds linearly to base level changes and can therefore be written in the form

$$Q_i(t) = Q_i^0 + Q_i'\left(H_b(t) - H_b(t_0)\right). \tag{17}$$

Here, Q_i^0 is the flux that occurs if the base level H_b remains constant ($H_b(t) = H_b(t_0)$), and Q_i' is the derivative of $Q_i(t)$ with regard to base level changes. Inserting Eq. (17) for the donors into Eq. (16) yields

$$s_i \frac{H_i(t) - H_i(t_0)}{\delta t} = s_i U_i - Q_i(t) + \sum_j Q_j^0 + \sum_j Q_j'\left(H_i(t) - H_i(t_0)\right) \tag{18}$$

and thus

$$Q_i(t) + \frac{\alpha_i}{\delta t}\left(H_i(t) - H_i(t_0)\right) = \beta_i \tag{19}$$

with the terms

$$\alpha_i = s_i - \delta t \sum_j Q_j' \quad \text{and} \quad \beta_i = s_i U_i + \sum_j Q_j^0 \tag{20}$$

introduced in order to keep the equations short. Similarly to the detachment-limited model, nodes without any donors act as boundaries within the domain. These nodes do not require any specific treatment except that the respective sums in Eqs. (18) and (20) are empty.

The channel slope at the node i is

$$S_i(t) = \frac{H_i(t) - H_b(t)}{d_i}, \tag{21}$$

where d_i is the distance between the nodes i and b. So the sediment flux is

$$Q_i(t) \;=\; KA_i^{m+1}\frac{H_i(t)-H_b(t)}{d_i} \tag{22}$$

according to Eq. (11) for $n = 1$. This leads to

$$H_i(t) = H_b(t) + \frac{d_i}{KA_i^{m+1}}Q_i(t). \tag{23}$$

Inserting this relation into Eq. (19) yields

$$Q_i(t) + \frac{\alpha_i}{\delta t}\left(\frac{d_i}{KA_i^{m+1}}Q_i(t) + H_b(t) - H_i(t_0)\right) = \beta_i, \tag{24}$$

which can be rearranged in the form

$$Q_i(t) = \frac{\alpha_i\left(H_i(t_0)-H_b(t)\right)+\beta_i\delta t}{\alpha_i\frac{d_i}{KA_i^{m+1}}+\delta t}. \tag{25}$$

Comparing this expression with Eq. (17) yields

$$Q_i^0 = \frac{\alpha_i\left(H_i(t_0)-H_b(t_0)\right)+\beta_i\delta t}{\alpha_i\frac{d_i}{KA_i^{m+1}}+\delta t} \tag{26}$$

and

$$Q_i' = -\frac{\alpha_i}{\alpha_i\frac{d_i}{KA_i^{m+1}}+\delta t}. \tag{27}$$

Equations (26) and (27) allow for the computation of Q_i^0 and Q_i' from the respective values of the donors and (because α_i and β_i depend on these) and from known elevation values at the time t_0. All values Q_i^0 and Q_i' can thus be computed successively in downstream direction. As the required order of the nodes is the same as for computing the catchment sizes A_i, it is most efficient to calculate Q_i^0 and Q_i' in the same sweep over the nodes where the catchment sizes are computed.

Once the values Q_i^0 and Q_i' have been computed for all nodes, the sediment flux $Q_i(t)$ can be computed using Eq. (17). This sediment flux is then used for computing the elevation $H_i(t)$ from Eq. (23). As these steps require the elevation of the flow target $H_b(t)$, they have to be performed successively in upstream order. This order is the same as used in the implicit scheme for detachment-limited erosion.

So the numerical scheme consists of three sweeps over the grid:

Sweep 1: Compute the flow directions b of all nodes. The nodes can be processed in any order.

Sweep 2: Compute the catchment size A and the properties Q^0 (Eq. 26) and Q' (Eq. 27) of all nodes. The nodes have to be processed in downstream order, e. g., by a recursive implementation. . This is implemented most conveniently in a recursive scheme with a function that computes the three above properties for each node. Before computing these values, the function checks which of the donors have already been treated and invokes itself for those donors that have not been considered before.

Table 1. Time complexity of the scheme for transport-limited erosion compared to the implicit scheme for detachment-limited erosion. CPU time was normalized to the total effort of one time step for detachment-limited erosion.

	Detachment limited		Transport limited	
	properties	CPU time (%)	properties	CPU time (%)
sweep 1	b	38	b	38
sweep 2	A	49	A, Q^0, Q'	54
sweep 3	H	13	Q, H	20
total		100		112

Sweep 3: Compute $Q(t)$ according to Eq. (17) and $H(t)$ from Eq. (23) for all nodes. The nodes must be processed in upstream order, ~~e.g.,~~ which is also performed conveniently by a recursive implementation. The principle is the same as in sweep 2 except that the flow target has to be considered instead of the donors.

The scheme is a direct scheme without any iterative component. The derivatives Q' are always negative (lower base level leads to a higher sediment flux), so that the properties α and thus the denominator in Eqs. (26) and (27) are always positive. So the scheme is unconditionally stable, and its time complexity is linear (~~$O(n)$~~$O(N)$) under all conditions.

The workflow with the three sweeps is basically the same as in the implicit scheme for detachment-limited erosion. The structure is the same without any extra loops, conditions or functions to be invoked. Additional effort only arises from floating-point operations. Table 1 provides an estimate of the time complexity compared to detachment-limited erosion. All results were obtained using the landform evolution model OpenLEM that was used in some previous studies (e.g., Robl et al., 2017; Wulf et al., 2019; Hergarten, 2020a), but has not been published explicitly. A regular 5000×5000 grid was used, and the results of several runs involving 100 to 1000 time steps were checked for consistency. The CPU time was normalized to the total effort of one time step for detachment-limited erosion. The difference in time complexity between both models is marginal.

With regard to memory complexity, the scheme presented here requires two additional variables per node, Q^0 and Q'. When performing the third sweep, one of them can be recycled for storing the original surface height $H(t_0)$ that is needed later when Eq. (17) is applied to the donors. The remaining variable can be used for storing the actual sediment flux $Q(t)$ in case it is needed later.

4 A numerical example

~~As comparing detachment-limited and~~ The transport-limited ~~erosion in detail would go beyond the scope of this study, only a simple example of steady-state topographies is given here. Investigating the temporal behaviorturned out to be quite complex in preliminary experiments and will be subject of further studies.~~ model proposed in Sect. 2 is equivalent to the detachment-limited model only for uniform erosion. Transient states are typically characterized by spatially variable erosion, so the two end members cannot cannot yield the same transient behavior. This result is, however, already clear from more general arguments

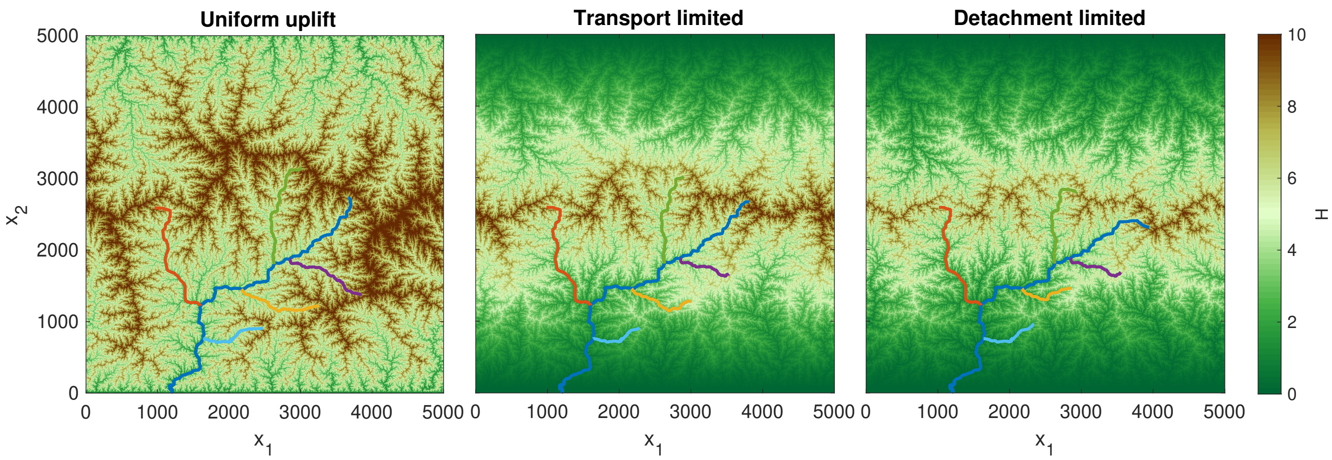

Figure 2. Equilibrium topographies for uniform uplift (left) and for a tent-shaped uplift pattern (middle and right). The color-coded rivers are the largest stream and its 5 largest tributaries in the topography for uniform uplift. They are referred to in Fig. 4.

since both end members are described by differential equations of different types (parabolic vs. hyperbolic) as discussed in Sect. 1.

The example presented here This section presents numerical example showing that non-uniform conditions result in strong differences between the two models even in a steady state. The example uses a square domain of 5000×5000 nodes. The northern and southern boundaries are kept at $H = 0$, while the two other boundaries are periodic. All horizontal lengths and areas are measured in terms of pixels. An exponent $m = 0.5$ was assumed, so that equilibrium rivers have a concavity index of $\theta = 0.5$ for the linear model ($n = 1$). The erodibility was set to $K = 1$.

Equilibrium topographies were computed by starting with small increments δt that are increased through time. At large δt, smaller random values of δt are used in each second step in order to avoid periodic oscillations between topographies with different flow patterns that prevent the topography from reaching a steady state.

An equilibrium topography obtained for uniform uplift $U = 1$ was used as a reference. This topography (Fig. 2, left) was generated by starting from a flat initial topography with a small random disturbance. As the transport-limited and the detachment-limited models are equivalent for uniform erosion, this topography is an equilibrium topography for both models.

As a simple non-uniform uplift pattern, tent-shaped uplift is considered. The maximum uplift rate $U = 1$ is achieved here in the middle between the northern and southern boundary ($x_2 = 2500$) and decreases linearly to zero towards the boundaries. In order to get similar flow patterns (Fig. 2), the equilibrium topography corresponding to constant uplift was used as an initial condition.

All equilibrium topographies were computed by starting with small increments δt that are increased through time when the number of changes in flow direction per time step is sufficiently small. This procedure is useful for generating steady-state topographies with similar large-scale flow patterns at a reasonable number of time steps. At large δt, smaller random values of

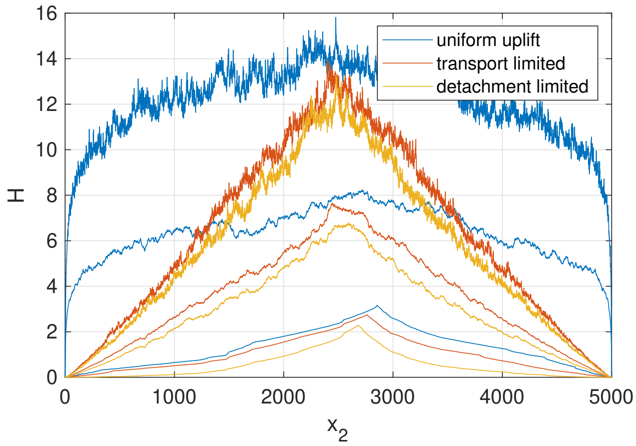

Figure 3. Swath profiles through the topographies shown in Fig. 2. The three lines of each color describe maximum, mean, and minimum elevation in east-west direction, i.e., over all values of x_1.

δt were used in each second step in order to avoid periodic oscillations between topographies with different flow patterns that prevent the topography from reaching a steady state.

The tent-shaped uplift pattern causes an overall increase in uplift in upstream direction at least for large rivers. This increase results in an upstream increase of steepness. As the steepness reflects the mean erosion rate of the upstream catchment (Eq. 15) instead of the local erosion rate for transport-limited erosion, it varies more gently with the uplift rate here than for detachment-limited erosion.

Figure 3 shows swath profiles through the three topographies. The maximum surface height (uppermost curve of the respective color) is dominated by the steep slopes at small catchment sizes. Since these depend on the local uplift rate in equilibrium, the maximum elevation roughly follows the tent-shaped uplift pattern with minor differences between transport-limited and detachment-limited erosion. The absolute difference between the two models is similar for maximum, mean, and minimum elevation, so it can be attributed to the different heights of large valleys, while local relief is similar.

The profiles of the large rivers marked in Fig. 2 are shown in Fig. 4. For a clearer representation, the longitudinal coordinate was χ transformed according to Eq. (7) with $A_0 = 1$. With the value $K = 1$ used here, equilibrium profiles follow a straight line $H = \chi$ at a uniform uplift rate $U = 1$. In turn, χ-transformed equilibrium profiles are concave if the uplift rate increases in upstream direction. This concavity is weaker for the transport-limited model than for the detachment-limited model as the local slope reflects the mean erosion rate of the upstream catchment, while it reflects the local erosion rate for detachment-limited erosion. In the upper part of the catchment, however, both turn into parallel straight lines. In the lower part of the catchment, the river profiles of the transport-limited model are steeper than those of the detachment-limited model because the river also has to carry away the material from the upper part with high erosion rates.

While the χ-transformed river profiles of the transport-limited model are more straight than for detachment-limited erosion, local collinearity of tributaries is lost. For detachment-limited erosion, profiles of tributaries start with the same slope as the

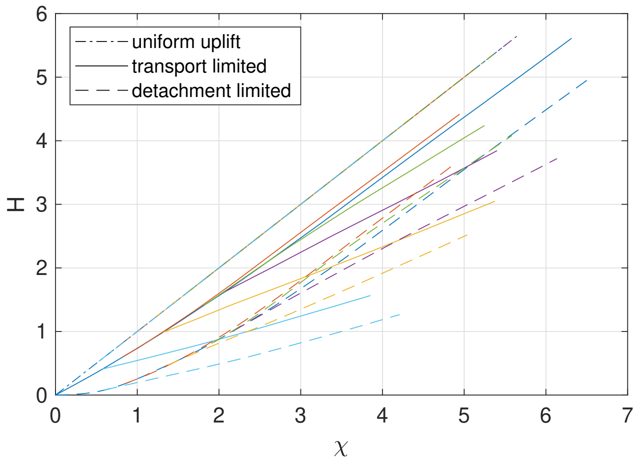

Figure 4. Longitudinal profiles of the rivers marked in Fig. 2 plotted in χ representation.

trunk stream and deviate more and more with increasing distance. In contrast, tributaries and the trunk stream may contribute different amounts of sediment per catchment size due to different mean erosion rates in their upstream catchments, which leads to different slopes immediately above the point of confluence in the transport-limited model. As a consequence, the capture of tributaries leads to stable knickpoints in the trunk stream for transport-limited erosion.

5 Combination with detachment-limited erosion

The numerical scheme described in Sect. 3 can be extended towards The most important lesson to be learned from this simple example concerns the estimation of the exponent n. Figure 5 shows the relation between the steepness index k_s and the erosion rate E, which is the same as the uplift rate U in a steady state. According to Eq. (14), the erosion rate is proportional to k_s^n in the detachment-limited erosion at least in two ways. First, it can be transferred to linear decline models. Second, the sum of two erosion processes can be considered where a sediment balance is taken into account only for a part of the eroded material, while the rest is immediately excavated. In both cases, however, only the linear version with regard to H ($n = 1$)can be implemented as a direct solver, while nonlinearity requires either a mixed scheme (some dependencies considered at t_0 instead of t)or an iterative treatment. regime. So comparing k_s at different locations exposed to different erosion is a common approach to estimate n (e.g., Lague, 2014).

4.1 Application to linear decline models

The general form of a linear decline model where the fluvial incision term is also linear reads $E = \phi S - \psi Q$, where S is channel slope and

The detachment-limited model with $K = 1$ and $n = 1$ reproduces the expected relation $E = k_s$, while this is not the case for the transport-limited model. Here the curve for the trunk stream (blue line) rather looks like a straight line with an offset. If we

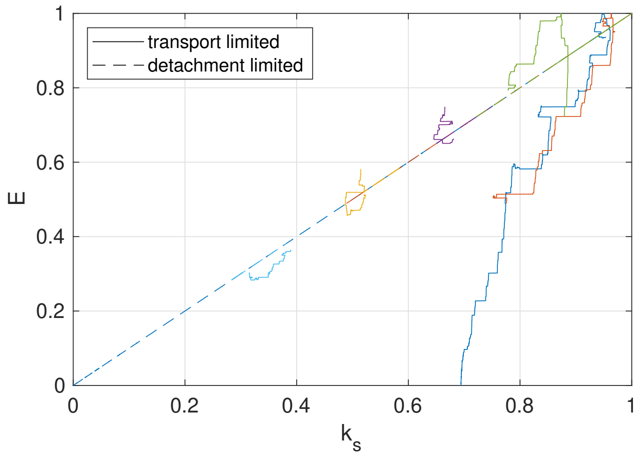

Figure 5. Erosion rate $E\ (=U)$ vs. steepness index k_s for the rivers marked in Fig. 2.

analyzed this curve without knowing that it originated from the transport-limited model, we would find that erosion starts at a threshold steepness index $k_s \approx 0.7$. If only a few points from this line were available, we would arrive at a nonlinear relation with $n > 1$. This is, however, an extreme example as $E = U = 0$ at the boundary, while the sediment flux from the domain requires a nonzero channel steepness. Qualitatively, the result would be similar in any situation where the uplift rate increases in upstream direction. If we interpret a long transport-limited river profile in terms of detachment-limited erosion, the exponent n would be systematically overestimated.

In turn, comparing the three tributaries that are predominantly oriented in east-west direction would yield an exponent close to the correct exponent $n = 1$ used here. The reason is that these tributaries are not subject to strong variations in uplift rate. All estimates reviewed by Lague (2014) suggest $n > 1$ except for one data set. This data set describes strike-parallel tributaries originally investigated by Kirby and Whipple (2001) where a re-analysis by Wobus et al. (2006) resulted in $n \approx 1$. This finding sheds new light on the apparent evidence for exponents $n > 1$ obtained from analyzing river profiles under non-uniform conditions. An unrecognized contribution of sediment transport may result in an overestimation of n here. This problem makes estimating the effective values of n even more difficult and deserves further consideration in the future.

5 Extension towards the linear decline model

Detachment-limited erosion and transport-limited erosion can be seen as end members of a more general framework. In particular, the extension of the detachment-limited model by sediment transport proposed by Davy and Lague (2009) is receiving growing interest in this context. Recently, Guerit et al. (2019) derived estimates of the sediment deposition parameter occurring in this model from analyzing natural and experimental topographies. The authors concluded that “natural landscapes seem to describe a continuum between the two modes with a preference for TL mode” (transport-limited mode) as already suggested by Davy and Lague (2009).

Whipple and Tucker (2002) already proposed the generic form of the model of Davy and Lague (2009) and coined the term "linear decline model". This concept starts from the detachment-limited model and assumes that the sediment flux reduces the ability of the river to erode. Assuming that the decrease in erosion rate is linear, this leads to the expression

$$E = f(A^\theta S) - \psi(A)Q \tag{28}$$

(from Eq. 4), where ψ is an arbitrary function.

In addition to Eq. (28), the sediment balance (Eq. 2 or the respective discrete form, Eq. 12) must be satisfied. Inserting Eq. (28) and the sediment balance into Eq. (1) yields a system of two coupled partial differential equations for the surface height H and the sediment flux $Q$is sediment flux .

The sediment balance can be written conveniently in integral form (Eq. 9). Combining this expression with Eqs. (1) and (28) yields a single integro-differential equation for the surface height,

$$\frac{\partial H}{\partial t} = U - f(A^\theta S) + \psi(A) \int \left(U - \frac{\partial H}{\partial t} \right) dA \tag{29}$$

instead of a system of two differential equations.

The functions ϕ detachment-limited end member corresponds to $\psi(A) = 0$. In this case, the two differential equations are decoupled, so the equation for H can be solved with computing Q. Approaching the detachment-limited end member is mathematically more complicated. This can be achieved by increasing f and ψ may depend on all properties except for surface heights (in such a way that $f \to \infty$ and consequently also channel slopes) in order to maintain the linearity. $\psi \to \infty$, while the ratio $\frac{f}{\psi}$ converges to a finite, nonzero value. Then, Eq. (28) turns into

$$Q = \frac{f(A^\theta S)}{\psi(A)}. \tag{30}$$

The resulting sediment flux Q defines the transport capacity.

If we request that equilibrium river profiles under uniform conditions are still consistent with Hack's findings (Eq. 3), the entire erosion rate defined by Eq. (28) must be a function of the product $A^\theta S$. Inserting Eq. (10) into Eq. (28) yields

$$E = \frac{f(A^\theta S)}{1 + A\psi(A)}. \tag{31}$$

So $\psi(A)$ must be inversely proportional to A,

$$\psi(A) = \frac{G}{A}, \tag{32}$$

with a nondimensional constant G. This is exactly the relation proposed by Davy and Lague (2009) (with Θ instead of G there). The second term in Eq. (28) turns into $\frac{GQ}{A}$, which was interpreted as deposition of sediments by Davy and Lague (2009).

The first term is either related to the equilibrium sediment flux equilibrium erosion rate under uniform conditions is

$$E = \frac{f(A^\theta S)}{1+G} = \frac{KA^m S^n}{1+G} \tag{33}$$

in this model. So sediment deposition effectively reduces the erosion rate by a factor of $1+G$ under uniform equilibrium conditions, which makes equilibrium profiles by a factor of $1+G$ steeper in the linear model (~~transport capacity) in~~ $n=1$) as already stated by Yuan et al. (2019). This effect can be compensated by rescaling the erodibility K by a factor of $1+G$, which modifies the model of Davy and Lague (2009) to

$$E=(1+G)\,KA^mS^n-G\frac{Q}{A}. \tag{34}$$

Both versions differ concerning the interpretation of the erodibility K. While it characterizes the process of detachment in the original model, it is interpreted as the fingerprint of spatially uniform erosion including sediment transport in the rescaled version. In contrast to the original version, the rescaled version also captures the transport-limited end member for $G\to\infty$ since

$$Q=\frac{(1+G)\,KA^mS^n-E}{\frac{G}{A}}\to KA^{m+1}S^n. \tag{35}$$

The linear decline model can be interpreted in several ways. If we define an equilibrium sediment flux by

$$Q_\mathrm{e}=\frac{f(A^\theta S)}{\psi(A)}, \tag{36}$$

Eq. (28) turns into

$$E=\psi(A)\,(Q_\mathrm{e}-Q)\,. \tag{37}$$

This is the undercapacity model (Kooi and Beaumont, 1994) written in terms of sediment flux instead of flux per unit width (Eq. 8).

The formulation of transport-limited erosion proposed in Sect. 2 allows for an alternative definition of a hybrid model that can also be interpreted as a linear decline model. Let us write the detachment-limited end member (Eq. 6) in the form

$$\frac{E}{K_\mathrm{d}}=A^mS^n, \tag{38}$$

and the ~~formulation of Kooi and Beaumont (1994) or to~~ transport-limited end member (Eq. 11) in the form

$$\frac{Q}{K_\mathrm{t}A}=A^mS^n. \tag{39}$$

In contrast to the ~~incision term in the concept proposed by Davy and Lague (2009).Accordingly, the second term is interpreted as deposition of particles by Davy and Lague (2009)~~ previous considerations, different symbols K_d and K_t are used here. As discussed above, their meaning is in principle the same, but there is no reason why the values should be the same under all conditions.

The simplest combination of both end members is assuming that the property $A^m S^n$ that is responsible for both detachment and transport is shared among the two processes, i.e.,

$$\frac{E}{K_\mathrm{d}} + \frac{Q}{K_\mathrm{t} A} = A^m S^n. \tag{40}$$

This model approaches the detachment-limited regime for zero sediment flux and the transport-limited regime for high sediment flux.

Equation (40) can also be written in the form

$$E = K_\mathrm{d} A^m S^n - \frac{K_\mathrm{d}}{K_\mathrm{t}} \frac{Q}{A}, \tag{41}$$

so

$$\psi(A) = \frac{K_\mathrm{d}}{K_\mathrm{t} A} \tag{42}$$

and

$$G = \frac{K_\mathrm{d}}{K_\mathrm{t}}. \tag{43}$$

The formulation defined by Eq. (40) could be called "shared stream power model". Compared to the concepts of detachment and deposition and the undercapacity model, is is rather a generic model. In turn, the formulation in terms of K_d and K_t may help to understand rivers passing different lithologies. Here we could expect that K_d shows a stronger variation than K_t, although the ability to transport material also depends on the characteristics of the sediments at the river bed.

The numerical scheme described in Sect. 3 can be extended towards the linear version of the linear decline model, i.e., if the first term in Eq. (28) is also linear in channel slope S ($n = 1$ in Eq. 6). The general form of this model reads

$$E = \phi(A) S - \psi(A) Q \tag{44}$$

with any functions ϕ and ψ. For the rescaled version of the model proposed by Davy and Lague (2009) and the shared stream power version, the two functions are

$$\phi(A) = \quad (1+G)\, K A^m \quad = K_\mathrm{d} A^m \tag{45}$$

$$\psi(A) = \quad \tfrac{G}{A} \quad = \frac{K_\mathrm{d}}{K_\mathrm{t} A}, \tag{46}$$

while the term $1 + G$ would not occur in the orginal version.

Inserting Eq. (44) into the general landform evolution model (Eq. 1) yields

$$\frac{\partial H_i}{\partial t} + \phi_i S_i - \psi_i Q_i - U_i = 0, \tag{47}$$

and after inserting difference quotients for time derivative and channel slope

$$\frac{H_i(t) - H_i(t_0)}{\delta t} + \phi_i \frac{H_i(t) - H_b(t)}{d_i} - \psi_i Q_i(t) - U_i = 0. \tag{48}$$

This equation can be rearranged in the form

$$H_i(t) = \frac{H_i(t_0) + \delta t \left(\frac{\phi_i}{d_i} H_b(t) + \psi_i Q_i + U_i \right)}{1 + \frac{\delta t \phi_i}{d_i}}. \tag{49}$$

Plugging this result into Eq. (19), rearranging the resulting equation to yield $Q_i(t)$, and comparing the obtained expression to Eq. (17) finally yields

$$Q_i^0 = \frac{\alpha_i \left(\frac{\phi_i}{d_i} \left(H_i(t_0) - H_b(t_0) \right) - U_i \right) + \beta_i \left(1 + \frac{\delta t \phi_i}{d_i} \right)}{\alpha_i \psi_i + 1 + \frac{\delta t \phi_i}{d_i}} \tag{50}$$

and

$$Q_i' = -\frac{\alpha_i \frac{\phi_i}{d_i}}{\alpha_i \psi_i + 1 + \frac{\delta t \phi_i}{d_i}}. \tag{51}$$

The scheme is very similar to that presented in Sect. 3 for transport-limited erosion. Equations (50) and (51) have to be used in sweep 2. Sweep 3 is now based on Eq. (49), while Eq. (17) is still used in its original form.

For the linear version of the formulation by Davy and Lague (2009), the expressions in Eq. (44) are

$$\phi = K A^m \quad \text{and} \quad \psi = \frac{G}{A}$$

where G is the coefficient of deposition as used by Yuan et al. (2019) for constant precipitation, equivalent to Θ used by Davy and Lague (2009). However, it was already stated by Yuan et al. (2019) that sediment deposition makes equilibrium river profiles by a factor of $1+G$ steeper. Assuming a uniform erosion rate and inserting $Q = EA$ into Eq. (44), it is immediately recognized that

$$(1+G)\,E = \phi S.$$

So the erosion rate is indeed by a factor of $1+G$ lower than without sediment deposition. Thus, Eq. (**??**) is appropriate if K is seen as a parameter of detachment-limited erosion. In turn, if K is interpreted as the fingerprint of spatially uniform erosion including sediment transport, the definition

$$\phi = (1+G)\,K A^m \quad \text{and} \quad \psi = \frac{G}{A}$$

would be more useful.

This version provides a model where the parameter G controls the transition from the detachment-limited model for $G = 0$ to the transport-limited model for $G \to \infty$ without changing the relation between K and river steepness. The numerical scheme

~~turns into the implicit scheme for detachment-limited erosion for $G = 0$, where the flux-related variables Q^0, Q', and $Q(t)$ are computed in each time step, but are not needed for computing $H(t)$. In the opposite limit ($G \to \infty$) the scheme approaches the scheme for the transport-limited case developed in Sect. 3.~~

Preliminary numerical tests revealed that the time complexity of this version is very close to the transport-limited case, while that of the iterative scheme proposed by Yuan et al. (2019) is close to ~~of~~ the detachment-limited case in each iteration step. In the first iteration, sweep 2 computes the catchment sizes here, while it integrates the upstream erosion rate to yield the sediment flux (Eq. 9) in subsequent iterations. Taking the values from Table 1, ~~this yields 112~~there would be a slight advantage of the iterative scheme (100 % vs. ~~162 %effort~~ 112 %) if the iterative scheme ~~requires only two iterations (less isimpossible~~ could be applied with a single iteration step. This is, however, not possible if the flow ~~directions change). So the~~ direction of any node has changed because the sediment flux Q is not available for the actual flow pattern then. In the best case, the iterative scheme requires two steps. According to the numerical tests of Yuan et al. (2019), this is achieved for small values of G in the order of magnitude of 0.01 at $n = 1$. This yields 112 % vs. 162 % effort, so the direct scheme is at least 30 % faster than the iterative procedure. ~~If the~~ The advantage of the direct scheme rapidly grows with increasing sediment transport. The data repository of the recent study of Guerit et al. (2019) found a median value of $G = 1.6$ for $n = 1$ by analyzing several natural river profiles. The iterative scheme requires ~~10 iterations ,~~ about 8 iterations at this value, resulting in an effort of 112 % vs. 534 %. So the direct scheme is ~~already 6 times faster and has the additional~~ almost 5 times faster under these conditions. Guerit et al. (2019) also reported on higher values of G, where the convergence of the iterative scheme would become very slow. In addition, the direct scheme has the advantage of an exact solution without the need for checking convergence.

6 Further extensions

6.1 Adding transport-limited and detachment-limited erosion

~~While linear decline models can bee seen as some kind of finite transport distance for all particles, another option is to~~ The models of the linear decline type discussed in Sect. 5 enforce a strict balance for the sediment flux for any nonzero function $\psi(A)$. However, we could also assume that a part of the eroded material is immediately excavated, while the rest is ~~described by~~ transported. This could be seen as a first step towards considering different particle sizes where one class of particles it so small that these will not be deposited. For simplicity, this version is elaborated only as an extension of the transport-limited model here, although a combination with the linear decline model is also possible. The linear version of this model reads

$$E = \mathrm{div}\boldsymbol{q} + \Gamma S, \tag{52}$$

or inserted into Eq. (1) and discretized in fully implicit form

$$s_i \frac{H_i(t) - H_i(t_0)}{\delta t} = s_i U_i - Q_i(t) + \sum_j Q_j(t) - s_i \Gamma_i S_i. \tag{53}$$

Here, Γ is any function that describes the excavation of material. Similarly to the functions ϕ and ψ used in the previous section, Γ may in principle depend on all properties except for surface heights in order to maintain the linearity. It may be tempting to use $\Gamma = \tilde{K}A^m$ in analogy to the detachment-limited model, where $\tilde{K}$ has the same meaning and physical unit as K. However, Eq. (52) combines a sediment balance with immediate excavation, which causes a scaling problem if rivers are considered as linear objects (Howard, 1994; Perron et al., 2008; Pelletier, 2010; Hergarten, 2020a). As a consequence, an additional rescaling factor depending on the pixel size must be introduced in the definition of Γ in order to avoid an artificial dependence of the results on the spatial resolution. Different approaches for this scaling factor are discussed in the above references.

Apart from this scaling problem, the numerical implementation is straightforward. Using Eq. (11), the last term in Eq. (53) can be expressed as

$$s_i\Gamma_i S_i = \frac{s_i\Gamma}{KA_i^{m+1}}Q_i(t). \tag{54}$$

This results in a factor $1+\frac{s_i\Gamma}{KA_i^{m+1}}$ in front of $Q_i(r)$ in Eq. (19). This factor propagates to the denominator of Eqs. (26) and (27), so that we finally arrive at

$$Q_i^0 = \frac{\alpha_i\left(H_i(t_0)-H_b(t_0)\right)+\beta_i\delta t}{\alpha_i\frac{d_i}{KA_i^{m+1}}+\left(1+\frac{s_i\Gamma}{KA_i^{m+1}}\right)\delta t} \tag{55}$$

and

$$Q_i' = -\frac{\alpha_i}{\alpha_i\frac{d_i}{KA_i^{\theta+1}}+\left(1+\frac{s_i\Gamma}{KA_i^{m+1}}\right)\delta t}\frac{\alpha_i}{\alpha_i\frac{d_i}{KA_i^{m+1}}+\left(1+\frac{s_i\Gamma}{KA_i^{m+1}}\right)\delta t}. \tag{56}$$

The further steps (Eqs. 17 and 23) remain the same.

7 ~~Further extensions and limitations~~

6.1 Hillslope diffusion

Linear diffusion (e.g., Culling, 1960) as the simplest model of hillslope erosion can be implemented more efficiently than in the detachment-limited model because the flux components in direction of the channel network can be integrated into the implicit scheme. If ~~D is the diffusivity and l_i~~ d_{ij} is the distance between the node i and a neighbor j and l_{ij} the length of the ~~edge between the~~ respective edge in a finite-volume representation, the diffusive flux in this direction is

$$Q_{ij}^{\text{diff}} = Dl_{ij}\frac{H_i-H_j}{d_{ij}}, \tag{57}$$

where D is the diffusivity. In contrast to the discrete divergence of the fluvial sediment flux density (Eq. 12), this simple expression is only valid if the edge is normal to the connecting line as it is the case, e.g., in a Voronoi discretization.

An implicit scheme for the flux components in direction of the channel network combined with an explicit scheme for the other components requires an additional variable B_i (where practically either Q_i^0 or Q_i' might be used) for the balance of the diffusive fluxes. For each node i~~and its~~, a loop over all neighbors j with $H_j < H_i$ except for the flow target b ~~, the diffusive flux in this direction is~~

~~$$Q_i^{\text{diff}} = Dl_i \frac{H_i - H_b}{d_i}.$$~~

~~Comparing this expression~~ is employed. The respective values Q_{ij} are added to B_j and subtracted from B_i. After considering all nodes i, the values $\frac{B_i}{s_i}$ are added to the uplift rates U_i. This part of the scheme captures the diffusive fluxes except for those in direction of the channel network. Comparing Eq. (57) to Eq. (22), it is easily recognized that the ~~term~~ diffusive flux from each node i to its flow target b can be included by replacing the term $KA_i^{\theta+1}$ ~~just has to be replaced by $KA_i^{\theta+1} + Dl_i$~~ by $KA_i^{\theta+1} + Dl_{ib}$ throughout the calculations of Sect. 3. ~~The other flux components still have to be handled by an explicit scheme, but~~

As this scheme is not fully implicit, the maximum time increment is still limited. However, as the flux component in flow direction is the largest among all, its partly implicit treatment improves the stability of the diffusion term and thus increases the maximum possible time increment.

~~However~~

7 Limitations

While the approach presented here is efficient and can be applied to a large class of problems, some limitations ~~of the approach~~ should also be mentioned.

First, any kind of sediment transport that transfers material from one site to more than one target site destroys the tree-like topology of sediment fluxes. Such processes are thus not compatible with the implicit scheme presented here. This applies ~~, e.g.,~~ to hillslope processes as well as to fluvial processes with multiple flow directions as implemented, e.g., in the model TTLEM (Campforts et al., 2017). However, the implicit scheme for detachment-limited erosion is subject to the same limitation.

Concerning numerics, nonlinearity is the only point where the approach suggested here falls behind the implicit scheme for detachment-limited erosion. The latter can be solved directly for $n = 1$ and for $n = 2$ (Hergarten, 2002), but can be treated by finding the roots of a scalar nonlinear equation at each point for any value of n. In contrast, nonlinearity can only be included in the approach proposed here either by treating the nonlinear terms in an explicit manner or to ~~use~~ employ an iteration.

When using the model with large time increments, it must be taken into account that all models in this field compute flow directions and changes in topography in separate steps. While the scheme itself is stable for arbitrarily large time increments, the applicability of large time increments is practically limited by changes in the flow directions. In particular, almost flat regions are susceptible to artifacts. Under erosion, unreasonable river networks may deeply incise at large time increments, and reorganization may take a long time afterwards. This problem also affects the detachment-limited model. In the aggradational regime, large rivers may even turn into weird ridges within a single, large time step. As the occurrence of such artifacts depends on the topography, it is even difficult to provide a rule of thumb for a reasonable maximum δt. Practically, tracking the number of changes in flow direction and adjusting δt so that the number of changes per time step does not exceed a given threshold provides a feasible criterion.

Finally, the treatment of lakes, i.e., local depressions in the topography, is a problem. In the detachment-limited model, local depressions result in negative channel slopes and thus in negative erosion rates without any specific treatment. However, these negative erosion rates can be cut off easily in the implicit scheme. In the transport-limited model, local depressions result in a sediment flux opposite to the flow direction. Erosion of dams may be too fast then, so that the lifetime of lakes may be too short. This effect cannot be fixed easily in the fully implicit scheme.

8 Conclusions

This study proposes a simple formulation of transport-limited fluvial erosion. This formulation can be immediately reconciled with the empirical results of Hack (1957) on longitudinal river profiles. The interpretation of Hack's findings as the fingerprint of spatially uniform erosion is equivalent for transport-limited erosion and for detachment-limited erosion where it has been widely used. In ~~particular, the main properties – concavity index and erodibility – are fully equivalent in both concepts.~~ turn, the behavior of both models differs if erosion is non-uniform.

As a main point, a new numerical scheme for treating transport-limited erosion with a fully implicit discretization in time was presented. It is a direct solver without any iteration and is unconditionally stable for arbitrarily large time increments. It is of linear time complexity (~~$\mathcal{O}(n)$~~$O(N)$) where the computing effort is marginally higher than for detachment-limited erosion. The scheme can also be applied to combined linear models of detachment-limited erosion and sediment transport such as the linear decline model. Here it also allows for approaching the transport-limited ~~ease~~ end member without any loss of performance and provides a numerical efficiency that is better than the iterative scheme suggested by Yuan et al. (2019).

Code and data availability. All codes and computed data can be downloaded from the FreiDok data repository (Hergarten, 2020b). The author is happy to assist interested readers in reproducing the results and performing subsequent research.

Author contributions. N/A

Competing interests. The author declares that there is no conflict of interest.

The author would like to thank Xiaoping Yuan, Wolfgang Schwanghart, Jean Braun, and an anonymous reviever for their constructive and encouraging comments. The article processing charge was funded by the Baden-Wuerttemberg Ministry of Science, Research and Art and the University of Freiburg in the funding programme Open Access Publishing.

References

Braun, J. and Willett, S. D.: A very efficient O(n), implicit and parallel method to solve the stream power equation governing fluvial incision and landscape evolution, Geomorphology, 180–181, 170–179, https://doi.org/10.1016/j.geomorph.2012.10.008, 2013.

Campforts, B., Schwanghart, W., and Govers, G.: Accurate simulation of transient landscape evolution by eliminating numerical diffusion: the TTLEM 1.0 model, Earth Surf. Dynam., 5, 47–66, https://doi.org/10.5194/esurf-5-47-2017, 2017.

Coulthard, T. J.: Landscape evolution models: a software review, Hydrol. Process., 15, 165–173, https://doi.org/10.1002/hyp.426, 2001.

Culling, W.: Analytical theory of erosion, J. Geol., 68, 336–344, https://doi.org/10.1086/626663, 1960.

Davy, P. and Lague, D.: Fluvial erosion/transport equation of landscape evolution models revisited, J. Geophys. Res. Earth Surf., 114, F03 007, https://doi.org/10.1029/2008JF001146, 2009.

Duvall, A. R. and Tucker, G. E.: Dynamic ridges and valleys in a strike-slip environment, J. Geophys. Res. Earth Surf., 120, 2016–2026, https://doi.org/10.1002/2015JF003618, 2015.

Eizenhöfer, P. R., McQuarrie, N., Shelef, E., and Ehlers, T. A.: Landscape response to lateral advection in convergent orogens over geologic time scales, J. Geophys. Res. Earth Surf., 124, 2056–2078, https://doi.org/10.1002/2019JF005100, 2019.

Ferrier, K. L., Perron, J. T., Mukhopadhyay, S., Rosener, M., Stock, J. D., Huppert, K. L., and Slosberg, M.: Covariation of climate and long-term erosion rates across a steep rainfall gradient on the Hawaiian island of Kaua´i, GSA Bull., 125, 1146–1163, https://doi.org/10.1130/B30726.1, 2013.

Flint, J. J.: Stream gradient as a function of order, magnitude, and discharge, Water Resour. Res., 10, 969–973, https://doi.org/10.1029/WR010i005p00969, 1974.

Goren, L., Fox, M., and Willett, S. D.: Tectonics from fluvial topography using formal linear inversion: Theory and applications to the Inyo Mountains, California, J. Geophys. Res. Earth Surf., 119, 1651–1681, https://doi.org/10.1002/2014JF003079, 2014.

Guerit, L., Yuan, X. P., Carretier, S., Bonnet, S., Rohais, S., Braun, J., and Rouby, D.: Fluvial landscape evolution controlled by the sediment deposition coefficient: Estimation from experimental and natural landscapes, Geology, 47, 853–856, https://doi.org/10.1130/G46356.1, 2019.

Hack, J. T.: Studies of longitudinal profiles in Virginia and Maryland, no. 294-B in US Geol. Survey Prof. Papers, US Government Printing Office, Washington D.C., https://doi.org/10.3133/pp294B, 1957.

Hackbusch, W.: Multi-Grid Methods and Applications, Springer, Berlin, Heidelberg, New York, https://doi.org/10.1007/978-3-662-02427-0, 1985.

Harel, M.-A., Mudd, S. M., and Attal, M.: Global analysis of the stream power law parameters based on worldwide 10Be denudation rates, Geomorphology, 268, 184–196, https://doi.org/10.1016/j.geomorph.2016.05.035, 2016.

Hergarten, S.: Self-Organized Criticality in Earth Systems, Springer, Berlin, Heidelberg, New York, https://doi.org/10.1007/978-3-662-04390-5, 2002.

Hergarten, S.: Rivers as linear elements in landform evolution models, Earth Surf. Dynam., 8, 367–377, https://doi.org/10.5194/esurf-8-367-2020, 2020a.

Hergarten, S.: Transport-limited fluvial erosion – simple formulation and efficient numerical treatment: codes and data, https://doi.org/10.6094/UNIFR/166660, 2020b.

Hergarten, S. and Neugebauer, H. J.: Self-organized critical drainage networks, Phys. Rev. Lett., 86, 2689–2692, https://doi.org/10.1103/PhysRevLett.86.2689, 2001.

Howard, A. D.: A detachment-limited model for drainage basin evolution, Water Resour. Res., 30, 2261–2285, https://doi.org/10.1029/94WR00757, 1994.

Kirby, E. and Whipple, K. X.: Quantifying differential rock uplift rates via stream profile analysis, Geology, 29, 415–418, https://doi.org/10.1130/0091-7613(2001)029<0415:QDRURV>2.0.CO;2, 2001.

Kooi, H. and Beaumont, C.: Escarpment evolution on high-elevation rifted margins: insights derived from a surface process model that combines diffusion, advection and reaction, J. Geophys. Res., 99, 12 191–12 209, 1994.

Lague, D.: The stream power river incision model: evidence, theory and beyond, Earth Surf. Process. Landforms, 39, 38–61, https://doi.org/10.1002/esp.3462, 2014.

Pelletier, J. D.: Minimizing the grid-resolution dependence of flow-routing algorithms for geomorphic applications, Geomorphology, 122, 91–98, https://doi.org/10.1016/j.geomorph.2010.06.001, 2010.

Perron, J. T. and Royden, L.: An integral approach to bedrock river profile analysis, Earth Surf. Process. Landforms, 38, 570–576, https://doi.org/10.1002/esp.3302, 2013.

Perron, J. T., Dietrich, W. E., and Kirchner, J. W.: Controls on the spacing of first-order valleys, J. Geophys. Res. Earth Surf., 113, F04 016, https://doi.org/10.1029/2007JF000977, 2008.

Robl, J., Hergarten, S., and Prasicek, G.: The topographic state of fluvially conditioned mountain ranges, Earth Sci. Rev., 168, 290–317, https://doi.org/10.1016/j.earscirev.2017.03.007, 2017.

Royden, L. and Perron, J. T.: Solutions of the stream power equation and application to the evolution of river longitudinal profiles, J. Geophys. Res. Earth Surf., 118, 497–518, https://doi.org/10.1002/jgrf.20031, 2013.

Theodoratos, N., Seybold, H., and Kirchner, J. W.: Scaling and similarity of a stream-power incision and linear diffusion landscape evolution model, Earth Surf. Dynam., 6, 779–808, https://doi.org/10.5194/esurf-6-779-2018, 2018.

Turowski, J. M.: Semi-alluvial channels and sediment-flux-driven bedrock erosion, in: Gravel-Bed Rivers, edited by Church, M., Biron, P., and Roy, A., chap. 29, pp. 399–418, John Wiley & Sons, Ltd, https://doi.org/10.1002/9781119952497.ch29, 2012.

van der Beek, P.: Modelling landscape evolution, in: Environmental Modelling: Finding Simplicity in Complexity, edited by Wainwright, J. and Mulligan, M., pp. 309–331, Wiley-Blackwell, Chichester, 2 edn., 2013.

Whipple, K. X. and Tucker, G. E.: Dynamics of the stream power river incision model: Implications for height limits of mountain ranges, landscape response time scales and research needs, J. Geophys. Res., 104, 17 661–17 674, https://doi.org/10.1029/1999JB900120, 1999.

Whipple, K. X. and Tucker, G. E.: Implications of sediment-flux-dependent river incision models for landscape evolution, J. Geophys. Res., 107, 2039, https://doi.org/10.1029/2000JB000044, 2002.

Whipple, K. X., DiBiase, R. A., and Crosby, B. T.: Bedrock rivers, in: Fluvial Geomorphology, edited by Shroder, J. and Wohl, E., vol. 9 of *Treatise on Geomorphology*, pp. 550–573, Academic Press, San Diego, CA, https://doi.org/10.1016/B978-0-12-374739-6.00226-8, 2013.

Willgoose, G.: Mathematical modeling of whole landscape evolution, Annu. Rev. Earth Planet. Sci., 33, 443–459, https://doi.org/10.1146/annurev.ea.33.092203.122610, 2005.

Willgoose, G., Bras, R. L., and Rodriguez-Iturbe, I.: A physical explanation of an observed link area-slope relationship, Water Resour. Res., 27, 1697–1702, https://doi.org/10.1029/91WR00937, 1991a.

Willgoose, G., Bras, R. L., and Rodriguez-Iturbe, I.: Results from a new model of river basin evolution, Earth Surf. Process. Landforms, 16, 237–254, https://doi.org/10.1002/esp.3290160305, 1991b.

Wobus, C., Whipple, K. X., Kirby, E., Snyder, N., Johnson, J., Spyropolou, K., Crosby, B., and Sheehan, D.: Tectonics from topography: Procedures, promise, and pitfalls, in: Tectonics, Climate, and Landscape Evolution, edited by Willett, S. D., Hovius, N., Brandon, M. T., and Fisher, D. M., vol. 398 of *GSA Special Papers*, pp. 55–74, Geological Society of America, Boulder, Washington, D.C., https://doi.org/10.1130/2006.2398(04), 2006.

Wulf, G., Hergarten, S., and Kenkmann, T.: Combined remote sensing analyses and landform evolution modeling reveal the terrestrial Bosumtwi impact structure as a Mars-like rampart crater, Earth Planet. Sci. Lett., 506, 209–220, https://doi.org/10.1016/j.epsl.2018.11.009, 2019.

Yuan, X. P., Braun, J., Guerit, L., Rouby, D., and Cordonnier, G.: A new efficient method to solve the stream power law model taking into account sediment deposition, J. Geophys. Res. Earth Surf., 124, https://doi.org/10.1029/2018JF004867, 2019.