Review of:

Particle size dynamics in abrading pebble populations

András A. Sipos, Gábor Domokos and János Török

General Comments

Very clever. Lovely.

This paper merits publication in *Earth Surface Dynamics* without too much ado. I suggest that revisions be focused on providing points of clarification for the *ESD* readership.

The centerpiece of the paper is its use of a Master equation, in the form of a Fokker-Planck equation, to describe the evolution of the probability distribution of particle states (sizes) in relation to abrasion due to particle collisions. A particularly interesting feature of the formulation is that the Fokker-Planck equation only involves advection. This occurs because the embedded kernel explicitly treats all possible changes in state associated with binary collisions involving each particle size with all other sizes, such that "diffusion" is unnecessary (but could be added depending on the physics involved). The formulation thus naturally leads to the ideas of *dispersion* versus *focusing* in describing the evolution of the form of the distribution of particle sizes, notably its variance — which represents a particularly interesting (and I think novel) outcome of the analysis. It is refreshing to see work that highlights how variability about the expected value is as important as the expected value in describing the dynamical behavior of the system.

Here I am compelled to offer a philosophical point regarding the value of the formalism provided by the Master equation and the Fokker-Planck equation. If we accept the veracity of the classical interpretation of probability (Hájek, 2012), then the Master equation is an element of physics (statistical mechanics) that is as close as it gets to the idea of "truth" that we normally reserve for the cumulative knowledge of mathematics. That is, it is a precise probabilistic ledger of admissible particle states and changes in these states. Then, the kernel within the Master equation (and the Fokker-Planck equation) contains the physics that govern the changes in state — just as, for example, the joint probability distribution of particle travel times and travel distances within the entrainment form of the Exner equation (a specialized Master equation) contains the essential physics of particle motions. I therefore view this paper as a particularly important contribution to the conversation on sediment particle comminution. The Master equation (Fokker-Planck equation) provides a wonderful framework. The path forward then involves further unfolding the physics of the kernel involved. The authors focus on a particle-mass-based kernel that is faithful to semi-empirical results regarding particle comminution (e.g., the exponential decline in the expected particle size). Perhaps eventually the physics can be elaborated to more explicitly incorporate, for example, particle collision energetics during downstream transport.

Specific Items

Figure 1 and accompanying text: The problem is conceptualized in terms of abrasion due entirely to particle-particle collisions. Because of this I anticipate the possibility of unease among some readers.

Specifically, each of the kernels, Eq. (5), Eq. (6) and Eq. (7), involves the masses X and Y of the *abraded* particle and the *abrader* particle. (The couplet in Eq. (1) clarifies that, with binary collisions, both particles are abraded and abrader particles.) This formulation thus essentially envisions binary collisions amongst a cohort of (continuously moving?) particles with a defined size distribution, and it excludes effects of particle-bed collisions. Inasmuch as a sediment bed is composed of particles and these are nominally included in the distribution of sizes, then strictly speaking all collisions involve particles with well-defined masses. But in natural and experimental settings, abrasion also involves particle-bedrock collisions. Moreover, inasmuch as bed particles are relatively immobile, then they act the same as bedrock with respect to an impacting particle. In this situation, the mass of the impacted "particle" (abraded? abrader?) is effectively infinite, such that the collision kernels are not meaningful. This issue becomes increasingly important as transport becomes rarefied, where effects of particle-bed collisions dominate over particle-particle collisions — akin to granular shear flows at high Knudsen number (Kumaran, 2005, 2006). Rarefied conditions represent an interesting limiting case. One might presume that the kernel involves only the mass of the impacting particle such that the Fokker-Planck equation takes on its ordinary form, but with inhomogeneous drift whose effect is focusing (depending on the value of r)?

That said, I raise these points to suggest the need for further clarity rather than as a criticism of the formulation. I view this work as an important, well-crafted brick designed to fit into the edifice of our understanding of particle comminution — *sensu* Forscher (1963). The selected kernels are a starting point. Effects of particle-bed impacts can wait for later.

Second, the authors comment on the idea that binary collisions constitute the basis of the collision kernel such that with large N, collective size evolution is driven by many binary collisions. I would add that, in fact, binary collisions are physically the correct choice. The formulation is continuous in time, and for this situation the probability that an individual particle experiences collisions with two or more particles simultaneously during an infinitesimal interval $\mathrm{d}t$ is vanishingly small if not zero. (Binary collisions of course are a foundational element of kinetic theory.)

Here is an idealized visual for what the formulation is describing. Envision a large container containing a great number N of particles (sort of like Figure 6d). We continuously, vigorously shake the container giving particle-particle collisions together with particle-container-wall collisions. For large N, effects of particle-particle collisions dominate over particle-container-wall collisions to produce abrasion. (This is directly reflected in the ODE form of Eq. (1), such that the formulation does not address downstream motion — which is entirely fine for starting the conversation.) Indeed, this visual is reminiscent of experimental abrasion studies that have in fact used containers of particles ("tumbling" experiments) to describe particle comminution. And, the shaking of a container with a great number of particles is essentially the basis of numerical simulations of granular gases (which have revolutionized this field).

Section 2.2: The authors are fully aware of the points made in the following comments, so these are merely offered in aiming at clarification of the presentation.

The quantities X and Y (and x and y) physically represent the same thing. In writing $f_t(t,x)$ on the left side of the Master equation, Eq. (4), this denotes that X is the size of interest — which experiences collisions with all other sizes Y via the collision kernel. That is, if X is the size of interest, then Y is a simple way of denoting all other sizes. It is then noted that, "contrary to the majority of Fokker-Planck models, spontaneous fragmentation is not allowed in our model, eliminating diffusion." First, the effect of fragmentation in this type of formulation would not actually be represented by diffusion. As previously alluded to, the formulation is number conserving, and fragmentation thus is not admissible. Letting $h = 1/N$ denote the small amount of probability "carried" by each of the great number N of particles, fragmentation of a particle of size X means that this amount of probability is instantaneously "lost" from the interval x to $x + \mathrm{d}x$ and partitioned into new amounts of probability $h = 1/(N-1+n)$ (where n is the number of new fragments) which instantaneously "appear" in new intervals $\mathrm{d}x$ associated with smaller values of x. That is, fragmentation is described by local sink and source terms. (If other formulations indeed treat this as a diffusive process, then I would be skeptical of them.) Second, the actual reason that diffusion is not involved is because the size-loss kernel is deterministic (rather than treated as a random variable) *and* because the integral within the brackets in Eq. (4) is over all y. This is the same as saying that the number of particles $Nf(t,x)\mathrm{d}x$ within the small interval x to $x + \mathrm{d}x$ during a small interval $\mathrm{d}t$ experiences binary collisions with all other possible sizes in proportion to their presence in the distribution. Because the kernel $\psi(x,y)$ is deterministic in x and y, all possible changes in size of particles with starting size x are explicitly accounted for so that a description of the rate of change in the variance of these changes in size — that is "diffusion" — is unnecessary. Hence the use of the words "dispersion" and "focusing" is accurate. If the kernel ψ instead is treated as a random variable, then diffusion becomes involved.

As an aside: Starting with the more basic form of the Master equation, I am wondering what the Kramers-Moyal expansion, or its counterpart, looks like in getting to Eq. (4). (There is no need to show this; I'm inspired to figure it out.)

Section 2.4, Lines 20–35: This presentation in relation to flow conditions is, for me, a bit cryptic. For example, the authors appeal to a linear velocity profile for laminar flows, so I immediately imagine the case of Couette flow and wonder about its relevance. I also am not sure what "cross-section" is being referred to. I would recommend further explanation. It seems that the key point is that the different flow conditions suggest values of r far from $r_{\mathrm{crit}} = 1/2$, leading to either dispersion or focusing behavior.

Section 3.3: This section is quite interesting. Based on a brief email exchange with Professor Domokos, I am suspecting that the authors will further elaborate the idea that the large particles ("outliers") in the numerical simulations involving finite N (in contrast to the continuum formulation) in a sense represent effects mentioned above — that particle abrasion in natural and experimental settings involves particle-bed/bedrock collisions, where the impacted "particle" (i.e., bedrock) has arbitrarily large mass. I also am envisioning situations in gravel bed streams where particularly large particles are present, consistent with the description in the Discussion.

Appendix A4: Once a Dirac function... always a Dirac function (for the compound kernel and regardless of the value of r). Of course! I love it!

I hope that my comments are useful to the authors.

djf

References

[1] Forscher, B. K.: Chaos in the Brickyard, *Science* **142**, 339 (1963).

[2] Hájek, A.: Interpretations of Probability, *The Stanford Encyclopedia of Philosophy* (Winter 2012 Edition), Edward N. Zalta (Ed.), http:// plato.stanford.edu/ archives/ win2012/ entries/ probability-interpret/ (2012).

[3] Kumaran, V.: Kinematic model for sheared granular flows in the high Knudsen number limit, *Physical Review Letters* **95**, 108001 (2005).

[4] Kumaran, V: Granular flow of rough particles in the high-Knudsen number limit, *Journal of Fluid Mechanics* **561**, 43–72 (2006).