# Response to Reviewer # 2

Kevin Pierce, Marwan Hassan, Rui Ferreira

Thank you for the detailed and constructive comments. We plan to revise the manuscript to address every point as we discuss below.

## 1 Major comments

1. " The paper is well written, clear and concise. The approach is sound and standard mathematical tools are briefly introduced before or after they are used, which helps to understand the main ideas behind technical derivations. Main equations however miss a detailed physical explanation, term by term, to be understood by the readership."

   Thanks - we took for granted that the basic structure of mobile-immobile models and the role of advection and diffusion terms is understood by the reader. To make the manuscript more approachable we will add more detailed term-by-term descriptions for major equations in the revised manuscript.

2. "The title should be more precise, several stochastic description of bedload having been already proposed."

   This is helpful advice. As you pointed out, one of the key contributions in the manuscript is the simultaneous treatment of motion-rest exchange and velocity fluctuations, so we will emphasize this in a revised title.

3. "A general concern is that the stochastic approach, although theoretically sound, is weakly linked to actual statistics of sediment transport by bedload, and thus the relevance of such complicated form of the bedload flux (eq 21!) is questionable for realistic transport conditions. In particular, there is no discussion on the actual values of Péclet number and the importance of considering both velocity fluctuations and entrainment/deposition as processes acting on similar time scales. There are considerable simplifications when decoupling both, so the authors should better point why such coupled approach is necessary. By doing so, the authors should also consider comparing their results with existing experimental or numerical data."

   We can estimate typical values of the Péclet number from the available experimental data and include these in the revised manuscript. Justifying the decoupling between motion-rest alternation and particle movement velocities is more difficult with existing data. One of us (Ferreira) has unpublished experimental data showing that the acceleration phase following entrainment is commonly more than ten times shorter than the period particles spend in motion. A similar conclusion is described in *Campagnol et al.* [2015]: here the acceleration phases following entrainment and preceding deposition are reported as commonly shorter than the durations of intermediate trajectories. We will include this citation in the revised manuscript to support our approximation. In any case our manuscript presents the first analytical model which describes particle entrainment and deposition simultaneously with particle motion – even if they are decoupled. For this reason we believe the manuscript will be important for any future modeling studies seeking to couple motion-rest exchange to the particle motion.

## 2 Minor comments

1. 12 : drop "really", and precise why /when fluctuations matter ?

Thanks, this is good advice. We will add content on the role of fluctuations in bedform initiation, the difficulty they present for transport prediction, and their emphasized magnitude in weak transport conditions.

2. 16 : What is a "classic" description ? Deterministic ?

   By "classic" we meant "of recognized and established value". We will change this to "deterministic" as it conveys the same point and avoids confusion.

3. 17-19 : I do not get the point here. The approach followed by the authors is also mainly kinematic in that no discussion is made on the forces (gravitational, drag, friction, collision,…) acting on particles.

   We are using modeled forces on particles to predict dynamics from Newtonian equations. This is distinctly different from CTRW approaches, for example, where modeled forces do not enter the description. The advantage is that we can change the modeled forces to represent friction, drag, etc. without changing the essential structure of the description. In particular, Eq. 7 follows for whatever choice of $F(v,t)$ in Eq. 5. For example, the results of *Lajeunesse et al.* [2017] follow for the choice $F(v) = \text{constant}$. Later, we plan to explore episodic collision forces as in *Pierce* [2021].

4. The original "probabilistic" description …

   We are happy to add "probabilistic". We are however not aware of an earlier model of grain-scale displacements than *Einstein* [1937].

5. 21 Later → replace by "more recently" (there were a lot a probabilistic studies between Einstein and Lisle)

   This is a fair point! We will modify.

6. 22 "by promoting his instantaneous steps to intervals of motion with constant velocity" I do not get the meaning of promoting hear.

   "Replacing" works just as well.

7. 75 - 85 No mention of Continuous Time Random Walks model is made. Authors should compare their approach with for instance [Schumer et al 2009]

   Our emphasis is on a more detailed treatment of the movement phase in a mobile-immobile model, so our stepping off point is from the two-state CTRWs like *Lajeunesse et al.* [2017], not the one-state CTRWs described in [*Schumer et al.*, 2009]. There are careful discussions on linkages between these models in our earlier works [*Pierce and Hassan*, 2020; *Pierce*, 2021]. We omitted this discussion here for brevity, but we can add the key CTRW references in the revision.

8. l115 : Better explain how this equation can be physically understood, notably the presence of k and ke with time derivatives.

   The structure $\partial_t(\partial_t + k)W = (\partial_t + k_E)W$ encodes an intermittency factor at long times. To see this, take small $t$: one can then neglect the low order time derivatives and integrate over time ($\int_{-\infty}^{t} dt$) to obtain $\partial_t W = LW$. From this we conclude that at short times, particles obey the Kramers equation. Now take large $t$. One now neglects the high order time derivatives to get $\partial_t W = \frac{k_E}{k} LW$. This is a Kramers equation with its velocity and position evolution *slowed* by the intermittency factor $k_E/k$ – the fraction of time the particle spends in motion. We will adapt this explanation to the revised manuscript.

9. l137 Is the overdamped approximation similar to adiabatic elimination of the fast variable ? A deeper discussion is needed here, notably the validity of such approximation with respect to typical bedload transport scales.

Yes, the analogy is to take the velocity as the fast variable. There are some papers by Van Kampen that discuss this point. We will add a more detailed discussion of the validity of the overdamped approximation using the discussion on bedload transport timescales in *Campagnol et al.* [2015].

10. l 143 : How does such expression compare with a spatio-temporal markov process, for instance eq 4.4 in Ancey & Heyman JFM 2014 ?

    The short answer is that they are not directly comparable because $P(x,t)$ is a single-particle density and not the particle activity which is considered in Ancey and Heyman. This said, there may be a way to relate these formulations by coarse graining our formulation. This will be something to investigate in the future.

11. l217 : Why would velocity fluctuation during motion decrease diffusion at small time scales ? I would have imagined the reverse.

    Thanks, this is a typo. As one can see in the Fig. 3b, velocity fluctuations *increase* the diffusion at short timescales compared to the analytical formula that neglects velocity fluctuations.

12. l230 Rewriting the Péclet in its usual form (diffusion time scale over advection time scale) would help understanding the transport process the authors are trying to characterize. In there definition of Peclet, the important length scale is the mean particle jump length. This should appear somewhere.

    Here the timescales are $\tau_a = V/k_D$ for advection and $\tau_d = \sqrt{2D/k_D}$ for diffusion – the displacements due to each process in the mean movement time $1/k_D$. Our Péclet number is then $Pe = (\tau_a/\tau_d)^2$. We'll add a mention of these timescales.

13. l264 : can you give an physical interpretation of why the flux is higher at the beginning ? Do we have a higher probability to sample particles in motion at short time scales ?

    The probability to sample particles in motion is constant through time given our chosen initial conditions in Appendix B. The flux is dominated by particle velocity fluctuations at short timescales, i.e. particles that transport much faster than the mean, whereas it is dominated by the ensemble mean velocity at long timescales, i.e. when the influence of velocity fluctuations from individual particles averages out. For this reason, the peak in the flux as $t \to 0$ grows as the Péclet number shrinks. We will add a statement to this effect in the revised manuscript.

14. Figure 4a : why is there a plateau between 1-100 s? Is the mean flux only dependent on Péclet and observation time ? If yes, can you make it appear clearly in eq 21. If not, what are the fixed parameter in this figure ? Can you compare with experimental/numerical (DEM) data ?

    We can infer from the manuscript equations that the *asymptotic* values of $\langle q(T)\rangle/q_0$ only depend on $Pe$ and $k_DT$, but this says nothing about the intermediate ("plateau") region, which may depend on the other parameters. We have

    $$\frac{q}{q_0} \sim \begin{cases} \sqrt{\frac{Pe}{2\pi k_D T}}, & T \ll (k_D Pe)^{-1} \\ 1, & T \gg (k_D Pe)^{-1} \end{cases}. \tag{1}$$

    We chose not to plot figure 4a using the dimensionless $k_DT$ because all curves then partially overlap, which obscures the patterns (like the emergence of a plateau) and makes for challenging visual comparison between the analytical and numerical results. We will clearly state which parameters are modified and which are held constant in a revised figure caption.

    At intermediate times, we suspect the plateau emerges (when the diffusion is relatively strong) because velocity fluctuations continue to control the flux, but both negative and positive excursions of velocity become relevant – Particles that are both slower and faster than the mean can cross the control surface. However we are not clear on this point, so we chose not to include it in the paper.

15. Figure 4b If the distribution is Poissonian, you should be able to rescale it by its mean and have a single time-independent distribution. Could you plot this ?

    Yes, we can certainly rescale, but we decided not to plot the distribution in this way in favor of demonstrating how the distribution width changes with observation time in Fig. 4b. We preferred to demonstrate in Fig. 4b how the *Einstein* [1950] theory gradually emerges as $T \to \infty$. Probably we can add a sentence indicating that the distribution collapses with the mean.

# References


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